Staple!

Name: ______________________________ **Ord:** ___ ___

Plex
MAA4402 2838

IndividualOP-V

Prof. JLF King
Wedn, 17Apr2024

PLEX IOP is due **2PM, Thurs., 25Apr2024**, slid *completely* under my office door, 402 LITTLE HALL. Print this sheet, which is "Page 1/N", and number your write-up as "page 2/N", "page 3/N" …"Page N/N".

V1: Short answer. Show no work.

___ ___ **a** Entire fnc H is non-zero on SCC S. In $\mathring{\mathsf{S}}$, our H has exactly two zeros, at $\mathbf{p} \neq \mathbf{q}$, with multiplicities $\mathsf{m},\mathsf{n} \in \mathbb{Z}_+$. Integral $\frac{1}{2\pi\mathbf{i}} \oint_{\mathsf{S}} \frac{z^k \cdot H'(z)}{H(z)} \mathrm{d}z =$ ______________________, for $k = 1, 2, 3, \ldots$. [*Hint:* Examine the proof of TAP and Rouché.]

___ ___ **b** Let $h(z) := \exp(\frac{1}{z}) \cdot \exp(\frac{1}{3z})$. Then $\operatorname{Res}_{z=0}(h(z)) =$ ____________. And $\operatorname{Res}_{z=0}(z \cdot h(z)) =$ ____________.

___ ___ **c** Consider entire function $G(z) := \sum_{n=0}^{\infty} a_n \cdot [z-8]^n$. Then

$$\operatorname{Res}_{z=0}\left(\frac{G(z)}{z^4}\right) = \sum_{n=K}^{\infty} a_n \cdot W_n, \quad \textit{where}$$

$K =$ ______ $\in \mathbb{N}$ and $W_n =$ __________________________.

[Number W_n is a fnc of n, and does not mention $a_0, a_1, a_2, \ldots$. You may use binomial coefficients in expressing W_n.]

Essays must be TYPED, *and* (preferably) *double-spaced. Use the* Print/Revise ⟲ *cycle to produce good, well thought out, essays.* *Do* **not** *restate the problem; just solve it.*

V2: Use a keyhole contour to carefully compute

$$J := \int_0^\infty \frac{x^{1/4}}{[x+1]^3} \, \mathrm{d}x,$$

showing all the steps. Prove that the asymptotic contribution of the integral along the two circles, is zero. Use the branch of Arg (as in class) that takes values in $[0, 2\pi)$. [*Hint:* As models, use the two examples in PlexNotes, as well as the GCIF to compute the appropriate residues.]

V3: With $\mathsf{U} := \operatorname{Sph}_1(0)$ the unit circle, $\mathbb{B} := \mathring{\mathsf{U}}$ the open ball, suppose h is holomorphic on $\widehat{\mathsf{U}}$, is non-constant, and $h(\mathsf{U}) \subset \mathsf{U}$. Prove that $h(\mathbb{B}) \supset \mathbb{B}$ as follows:

i Use Max-modulus Principle (and…) to prove $\exists p \in \mathbb{B}$ with $h(p) = 0$. [*FTSOContradiction…*]

ii Given target $\tau \in \mathbb{B}$, use Rouché to prove $\exists \mathbf{q} \in \mathbb{B}$ with $h(\mathbf{q}) = \tau$.

V4: For $N = 4, 5, 6, \ldots$, define open annulus *and polynomial*

$$\mathbf{A}_N := \left\{ z \in \mathbb{C} \;\middle|\; N < |z| < N+1 \right\},$$

$$T_N(z) := z^N + Nz + [2N^2 + N^N].$$

Polynomial $T_N()$ has ____________________ roots in $\mathbf{A}_N$.

Prove your result, using Rouché's thm, carefully specifying what contours you are using, giving a detailed, complete argument establishing the inequalities you need. [In your essay, use $\mathsf{C}_k := \operatorname{Sph}_k(0)$ for the k^{th}-circle.]

Provide good, LARGE, *Labeled* pictures of the annulus, the contours and T_N-zeros, at least for $N=4$ and $N=5$.

Something extra?: What happens for $N = 1, 2, 3$? *Can you*: Generalize the problem in a mathematically interesting way? Give me more information of the locations of the roots, as a fnc of N?

V5: For all $z \in \mathbb{C}$, entire fnc f has

$$\left|\operatorname{Re}(f(z)) - \operatorname{Im}(f(z))\right| \leq 4.$$

Prove that f is constant. [Liouville's thm is useful.]

V6: Function f is ***nice*** if f is analytic on open ball $\mathbb{B} := \operatorname{Bal}_1(0)$, *and* $f(0) = 3$. A radius $0<r<1$ is ***good*** if *some* point w on circle $\mathsf{C}_r := \operatorname{Sph}_r(0)$ satisfies $|f(w)| = 3$.

i Produce a *specific* nice fnc g with *bad* radius $b<1$.

ii *Main problem:* Prove each nice fnc h has some *positive* radius $\boldsymbol{\rho}<1$ for which: *Every* radius $r<\boldsymbol{\rho}$ is good. [*Hint:* Use ideas related to the Maximum-modulus principle. You may want the Intermediate-value theorem applied to some real-valued function.]

iii Create a *continuous* $\varphi:\mathbb{B}\to\mathbb{C}$ with $\varphi(0) = 3$ for which *every* positive radius is **bad**.

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor (or his colleague)."*

Ord:
____ ___ ___

Folks, I've had a great time learning Complex Analysis with you. Stop by in future semesters for Math/chess/pickleball. Cheers, Prof. (Per)PLEX*-ed*