Staple!

Ord: ___ ___

Linear Algebra
MAS4105 *3247*

Practice-T

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T1: Show no work.

___ ___ a The author(s) of our textbook is/are

⌞. . . . .⌟.

___ ___ b Matrix $\mathsf{P} \in \mathrm{MAT}_{5\times5}(\mathbb{R})$ is ***positive definite*** precisely means that ⌞. . . . .⌟

⌞. . . . .⌟

⌞. . . . .⌟.

___ ___ c Perm $\nu := [5, 7, 2, 9, 6, 8, 4, 1, 3]$ has $\mathrm{Sgn}(\nu) =$ **+1** [✓ **−1**].

Perm-sign: This ν switches a 3-block with a 5-block, hence can be realized with $3 \cdot 5 = 15$ transpositions, hence is an odd perm.

Alternatively, since $3 \perp 8$, this block-switch makes ν a single 8-cycle, so the number of even-len-cycles [***EL-cycles***] is 1. Thus $\mathrm{Sgn}(\nu) = [-1]^1 = -1$.

In CCN [Canonical Cycle Notation],

$$\nu = \circlearrowright 8 \to 5 \to 2 \to 7 \to 4 \to 1 \to 6 \to 3 \circlearrowleft .$$

___ ___ d Let $\mathbb{S}_8$ denote the set of permutations of $[1 .. 8]$. For an 8×8 matrix $\mathsf{M} = [\beta_{i,j}]_{i,j}$, write *the Leibniz formula for determinant*

$$\mathrm{Det}(\mathsf{M}) = \sum_{\llcorner \dots\dots \lrcorner} \Big[\qquad\qquad\qquad \Big].$$

⌞. . . . .⌟

e Apply Cramer's Rule to write x_1 as a *rational function*,

$x_1 =$ ...., of variables $A, G, R, T, U, \alpha, \beta, \gamma$, where

$$\begin{bmatrix} R & U & G \\ 0 & A & 0 \\ 0 & 0 & T \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

f An example of a 3×3 $\mathbb{R}$-matrix with min-poly $\Upsilon_{\mathsf{A}}(x) = [x+5]^2$, is $\mathsf{A} =$ ....

—— —— $\boxed{g}$ Let $\mathsf{P}\colon\mathbb{R}^2\to\mathbb{R}^2$ be orthogonal projection on the θ-angle line. [Picture on blackboard.] In terms of numbers $\mathbf{c} := \cos(\theta)$ and $\mathbf{s} := \sin(\theta)$, then,

w.r.t the std-basis, $[\![\mathsf{P}]\!]^{\mathcal{E}}_{\mathcal{E}}$ equals $\left[\begin{array}{c|c} \qquad & \qquad \\ \hline \qquad & \qquad \end{array}\right]$.

Projection Matrix: Rotate the line by $-\theta$, orthoproject on the x-axis, then rotate back. So $[\![\mathsf{P}]\!]^{\mathcal{E}}_{\mathcal{E}}$ is the product

$$\begin{bmatrix}\mathbf{c} & -\mathbf{s}\\ \mathbf{s} & \mathbf{c}\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}\mathbf{c} & \mathbf{s}\\ -\mathbf{s} & \mathbf{c}\end{bmatrix} = \begin{bmatrix}\mathbf{c} & 0\\ \mathbf{s} & 0\end{bmatrix}\cdot\begin{bmatrix}\mathbf{c} & \mathbf{s}\\ -\mathbf{s} & \mathbf{c}\end{bmatrix} \overset{\text{note}}{=\!=} \begin{bmatrix}\mathbf{cc} & \mathbf{cs}\\ \mathbf{sc} & \mathbf{ss}\end{bmatrix}.$$

This matrix is indeed a rank-1 (as the middle product shows).

—— —— $\boxed{h}$ A map $f\colon\mathbf{V}\times\mathbf{E}\to\mathbf{W}$ (where $\mathbf{V}, \mathbf{E}, \mathbf{W}$ are $\mathbb{R}$-vectorspaces) is ***bilinear*** if $\forall\,\mathbf{v}_1,\mathbf{v}_2 \in \mathbf{V}$, $\forall\,\mathbf{e}_1,\mathbf{e}_2 \in \mathbf{E}$ and

$\forall$ ⌞. . . . .⌟ : ⌞. . . . .⌟

and ⌞. . . . .⌟.

A map $\langle\cdot,\cdot\rangle$ from $\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ is an ***inner product*** if

I1: ⌞. . . . .⌟.

I2: ⌞. . . . .⌟.

I3: ⌞. . . . .⌟.

i Define $\mathbf{b}_0 := \left[\begin{smallmatrix}-1\\4\end{smallmatrix}\right]$ and $\mathbf{b}_1 := \left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]$. Then inner-product (on 2×1 colvecs) $\langle \mathbf{u}, \mathbf{w}\rangle := \mathbf{u}^{\mathrm{t}}\,\mathsf{P}\,\mathbf{w}$, makes $\mathcal{B} := \{\mathbf{b}_0, \mathbf{b}_1\}$ into an orthonormal basis, for $2{\times}2$ pos-definite matrix $\mathsf{P} =$ ________________________.

I'm positive that I'm DEFINITE!. The numbers get too big for the following problem to be on an in-class exam.

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  Given colvecs b0 and b1 (defined below),
we compute the unique
positive-definite matrix P such that
the inner-product  (acting on colvecs)

    (x0, x1) |->   TPose(x0) * P * x1

makes {b0,b1} an ortho-normal basis of |R^3.


                LISP CODE
;; For testing purposes, define the std basis:
(setq   e0 (mat-make-colvec 1 0)   e1 (mat-make-colvec 0 1))
    [ 1 ]     [ 0 ]
    [ 0 ] ,   [ 1 ]

(setq b0 (mat-make-colvec -1 4)     b1 (mat-make-colvec 5 2))
    [ -1 ]    [ 5 ]
    [  4 ] ,  [ 2 ]

;; This will realize B-orthonormal IP, once I compute matrix P.
(defun IP-by-P (x0 x1) (mat-E (mat-mul (mat-tpose x0) P x1) 0 0))


;; Computing 2x2 matrix P:
(setq Id-ToE-FrB (mat-Horiz-concat b0 b1))
    [ -1  5 ]
    [  4  2 ]

(setq Id-ToB-FrE (mat-inverse? Id-ToE-FrB))
    1      [ -2   5 ]
   ---  *  [        ]
   22      [  4   1 ]

;; Let's check that we have the correct CoB matrix.
(mat-mul Id-ToB-FrE  b0)   =>  [ 1 ]
                               [ 0 ]

(mat-mul Id-ToB-FrE  b1)   =>  [ 0 ]
                               [ 1 ]


(setq P (mat-mul (mat-tpose Id-ToB-FrE) Id-ToB-FrE))
      1          [ 10   -3 ]
  --------   *   [         ]
  2 * 11^2       [ -3   13 ]

(IP-by-P b0 b1)  =>  0

(IP-by-P b0 b0)  =>  1

(IP-by-P b1 b1)  =>  1
```

Cool. . . ◆

OYOP: Essays: *Write on every* ***second*** *line, so that I can easily write between the lines.*

T2: Linear trn $\mathsf{T}\colon\mathbf{V}\to\mathbf{V}$ has T-invariant subspace $\mathbf{W}\subset\mathbf{V}$ [i.e $\mathsf{T}(\mathbf{W})\subset\mathbf{W}$]. Vectors $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors, with eigenvalues 5 and 2, respectively. Further, sum $\mathbf{u}+\mathbf{v}\in\mathbf{W}$.

Prove that $\boxed{\mathbf{u}\in\mathbf{W}}$. [*Hint:* Review our proof that the family of eigenspaces of a lin-trn is an LI-family. This is Eigenspace LI theorem on P.22 of "LinAlg-Notes (pdf)" on our webpage.]

[*Hint:* On P.322 of our textbook, see problem #23.]

T3: Fix a lin.trn $\mathsf{T}\colon\mathbf{X}\to\mathbf{X}$, where $\mathbf{X}$ is a (possibly ∞-dim'al) vectorspace.

.1 Subspace $\mathbf{U}\subset\mathbf{X}$ is T-***invariant*** if

.2 Consider a collection $\mathcal{C} := \{\mathbf{U}_1, \mathbf{U}_2, \ldots\}$ of T-invariant subspaces. Prove that intersection $\mathbf{W} := \bigcap_{n=1}^{\infty}\mathbf{U}_n$ is an **X**-subspace, <u>and</u> *is* T-*invariant*.

End of Practice-T

NAME: Ord:

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: *Energetic Student*