Generalized parentheses. **Common:**
( ) *Parentheses.* [ ] *Brackets.* ⟨ ⟩ *Angle-brackets.* { } *Braces.* | | *Vertical bars* (abs.value, cardinality).

Less common: ⌊ ⌋ *Floor.* ⌈ ⌉ *Ceiling.* ‖ ‖ *Norm-of.* ⟦ ⟧ *Double-bracket.* ⟪ ⟫ *Double-angle-bracket.*

Number Sets. Expression $k \in \mathbb{N}$ [read as "k is an element of $\mathbb{N}$" or "k in $\mathbb{N}$"] means that k is a natural number; a ***natnum***. Expression $\mathbb{N} \ni k$ [read as "$\mathbb{N}$ ***owns*** k"] is a synonym for $k \in \mathbb{N}$.

$\mathbb{N}$ = natural numbers = $\{0, 1, 2, \dots\}$.

$\mathbb{Z}$ = integers = $\{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the ***posints***, use $\mathbb{Z}_+$. Use $\mathbb{Z}_-$ for the negative integers, the ***negints***.

$\mathbb{Q}$ = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use $\mathbb{Q}_+$ for the positive rationals and $\mathbb{Q}_-$ for the negative rationals.

$\mathbb{R}$ = reals. The ***posreals*** $\mathbb{R}_+$ and the ***negreals*** $\mathbb{R}_-$.

$\mathbb{C}$ = complex numbers, also called the ***complexes***.

For $\omega \in \mathbb{C}$, let "$\omega > 5$" mean "ω *is* ***real*** *and* $\omega > 5$". [Use the same convention for $\geq, <, \leq$, and also if 5 is replaced by any real number.]

Use $\overline{\mathbb{R}} = [-\infty, +\infty] := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, the ***extended reals***.

An "***interval of integers***" $[b \,..\, c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[\mathrm{e} \,..\, 2\pi] = \{3, 4, 5, 6\} = [3 \,..\, 6] = (2 \,..\, 6]$. We allow b and c to be $\pm\infty$; so $(-\infty \,..\, -1]$ is $\mathbb{Z}_-$. And $[-\infty \,..\, -1]$, is $\{-\infty\} \cup \mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2\mathrm{i}| = \sqrt{29}$.

Mathematical objects. Seq: '*sequence*'. poly(s): '*polynomial(s)*'. irred: '*irreducible*'. Coeff: '*coefficient*' and var(s): '*variable(s)*' and parm(s): '*parameter(s)*'. Expr.: '*expression*'. Fnc: '*function*' (so ratfnc: means rational function, a ratio of polynomials). trnfn: '*transformation*'. cty: '*continuity*'. cts: '*continuous*'. diff'able: '*differentiable*'. CoV: '*Change-of-Variable*'. CoI: '*Constant of Integration*'. LoI: '*Limit(s) of Integration*'. RoC: '*Radius of Convergence*'.

Soln: '*Solution*'. Thm: '*Theorem*'. Prop'n: '*Proposition*'. CEX: '*Counterexample*'. eqn: '*equation*'. RhS: '*RightHand side*' of an eqn or inequality. LhS: '*lefthand side*'. Sqrt or Sqroot: '*square-root*', e.g, "the sqroot of 16 is 4". Ptn: '*partition*', <u>*but*</u> pt: '*point*' as in "a fixed-pt of a map".

FTC: '*Fund. Thm of Calculus*'. IVT: '*intermediate-Value Thm*'. MVT: '*Mean-Value Thm*'.

The ***logarithm*** function, defined for $x>0$, is $\log(x) := \int_1^x \frac{\mathrm{d}v}{v}$. Its inverse-fnc is $\exp()$.

For $x>0$, then, $\exp(\log(x)) = x = \mathrm{e}^{\log(x)}$. For real t, naturally, $\log(\exp(t)) = t = \log(\mathrm{e}^t)$.

PolyExp: '*Polynomial-times-exponential*', e.g, $[3 + t^2] \cdot \mathrm{e}^{4t}$. PolyExp-sum: '*Sum of polyexps*'. E.g, $f(t) := 3t\mathrm{e}^{2t} + [t^2] \cdot \mathrm{e}^t$ is a polyexp-sum.

Phrases. WLOG: '*Without loss of generality*'. IFF: '*if and only if*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. And ※="Contradiction".

IST: '*It Suffices To*', as in ISTShow, ISTExhibit.

Use w.r.t: '*with respect to*' and s.t: '*such that*'.

Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. inter alia: '*among other things*'. QED: *quod erat demonstrandum*, meaning "end of proof".

Sequence notation. A *sequence* $\vec{\mathbf{x}}$ abbreviates $(x_0, x_1, x_2, x_3, \dots)$. For a set Ω, expression "$\vec{\mathbf{x}} \subset \Omega$" means $[\forall n\colon x_n \in \Omega]$. Use $\mathrm{Tail}_N(\vec{\mathbf{x}})$ for the subsequence

$$(x_N,\, x_{N+1},\, x_{N+2}, \dots)$$

of $\vec{\mathbf{x}}$. Given a fnc $f\colon\Omega\to\Lambda$ and an Ω-sequence $\vec{\mathbf{x}}$, let $f(\vec{\mathbf{x}})$ be the Λ-sequence $(f(x_1), f(x_2), f(x_2), \dots)$.

Suppose Ω has an addition and multiplication. For Ω-seqs $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$, then, let $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ be the sequence whose n^{th} member is $x_n + y_n$. I.e

$$\vec{\mathbf{x}} + \vec{\mathbf{y}} \quad = \quad [n \mapsto [x_n + y_n]] \,.$$

Similarly, $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}}$ denotes seq $[n \mapsto [x_n \cdot y_n]]$.

LinA [2023t] quizzes

These count!

(Recall, the lowest MQ score is dropped. In consequence, there is *no make-up* for the first missed MQ.)

Q1: Tue. 05 Sep Line $y = Mx + B$ is orthogonal to $y = \frac{1}{5}x + 2$ and owns $(4, 10)$. So $M = -5$ and $B = 30$.

Soln. Line $y = \frac{1}{5}x + 2$ has slope $\frac{1}{5}$. Its negative reciprocal, value -5, is thus the slope of the orthogonal line.

Consequently, $\boxed{-5x + B = y}$ is the form of the orthogonal line; this, for an unknown B.

Plugging in $(4, 10)$ for (x, y) shows that $B = 30$. ♦

Values x-____, y-____, where $\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

Soln. Computing, $x = 2\cdot 2 + [-3]\cdot 5 \overset{\text{note}}{=\!=} -11$. Similarly, $y = 9$. ♦

Q2: Wed. 06 Sep Matrix-product $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 & w \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2w-3 \\ 38 & 4w+5 \end{bmatrix}$.

The ***trace*** of $\begin{bmatrix} -3/2 & -5/2 \\ 7/2 & 9/2 \end{bmatrix}$ equals ________.

Soln. The trace of a non-square matrix is *DNE*, Does Not Exist. For M a $N \times N$ matrix, $\text{Trace}(\mathsf{M}) = \text{Tr}(\mathsf{M})$ is the sum of the elements on the main diagonal.

Above, the trace equals $\frac{-3}{2} + \frac{9}{2} \overset{\text{note}}{=\!=} 3$. ♦

Q3: Mon. 11 Sep Blanks$\in\mathbb{R}$. So $\frac{1}{2+3\mathbf{i}} =$ _____ $+\mathbf{i}\cdot$ [_____].

Thus $\text{Im}\left(\frac{5-\mathbf{i}}{2+3\mathbf{i}}\right) =$ ________.

By the way, $|5 - 3\mathbf{i}| =$ ________.

Soln: $\mathbb{C}$-arithmetic. With $d := 2 + 3\mathbf{i}$, note

$$\frac{1}{d} = \frac{1}{d}\cdot\frac{\overline{d}}{\overline{d}} = \overline{d}\Big/|d|^2.$$

Since $|d|^2 = 2^2 + 3^2 = 13$, we have $\frac{1}{d} = \frac{\overline{d}}{13} = \frac{2}{13} + \frac{-3}{13}\mathbf{i}$.

Writing $\frac{1}{d} = B + C\mathbf{i}$ with B,C real, then,

$$B = \frac{2}{13} \quad \textit{and} \quad C = \frac{-3}{13}.$$

Letting $n := 5 - \mathbf{i}$, the above shows that

$$\frac{n}{d} = \frac{n\cdot\overline{d}}{|d|^2} = \frac{n\cdot\overline{d}}{13}.$$

Hence

$$\text{Im}\left(\frac{n}{d}\right) = \text{Im}(n\cdot\overline{d})/13,$$

since $13 \in \mathbb{R}$. Finally,

$$\begin{aligned} \text{Im}(n\cdot\overline{d}) &= \text{Re}(n)\cdot\text{Im}(\overline{d}) + \text{Im}(n)\cdot\text{Re}(\overline{d}) \\ &= 5\cdot[-3] + [-1]\cdot 2 = -17. \end{aligned}$$

Thus

$$\text{Im}\left(\frac{n}{d}\right) = \frac{-17}{13}.$$

Lastly, $|5 - 3\mathbf{i}|^2 = 5^2 + 3^2 = 34$. Consequently, $|5 - 3\mathbf{i}| = \sqrt{34}$.

Q4: Tue. 12 Sep !May lightning ↯ strike this table! (I.e, please fill in.)

n	r_n	q_n	s_n	t_n
0	81	—	**1**	**0**
1	66		**0**	**1**
2				
3				
4				
5				

So GCD$(81, 66)$= .... =[......... $\cdot\, 81$] + [......... $\cdot\, 66$].

LBolt Soln:

```
% (use-ring Integer-ring)

% (lightning 81 66)

      n:    r_n    q_n    s_n    t_n
    /----------------------------\
      0:     81     --      1      0
      1:     66      1      0      1
      2:     15      4      1     -1
      3:      6      2     -4      5
      4:      3      2      9    -11
      5:      0  Infty    -22     27
    \____________________________/

    Rename r4,s4,t4  to  GCD,S,T.

    Note     GCD   =   r0 * S   +   r1 * T,   i.e
               3   =   81 * 9   +   66 *[-11].
```
♦

Q5: Wed. 13 Sep Over field $\mathbb{Z}_{53}$, consider matrix $\mathsf{M} := \begin{bmatrix} 21 & 3 \\ 4 & 5 \end{bmatrix}$.

Its determinant is $\mathrm{Det}(\mathsf{M}) =$ ________ $\in [0..53)$, with

mod-53 reciprocal $\frac{1}{\mathrm{Det}(\mathsf{M})} =$ ________ $\in [0..53)$.

2×2 matrix determinant.

```
:: (set-ring-modulus (setq Prime 53))
:: (use-ring Mod-M-ring)
:: (setq M (mat-make-square 21 3 4 5))
    [  21  3  ]
    [   4  5  ]
```

So $\mathrm{Det}(\mathsf{M}) = 21{\cdot}5 - 3{\cdot}4 = 93 \equiv_{53} 40$.

As $40 \perp 53$ ("40 *is coprime to* 53") value 40 has a mod-53 reciprocal, which we can compute via LBolt. Recall the row property $\boxed{\forall n\colon\ r_n = [s_n \cdot 53] + [t_n \cdot 40]}$ of LBolt. Taking this mod-53 gives

$$\forall n\colon \quad r_n \equiv_{53} t_n \cdot 40\,.$$

Consequently, *we do not need to compute the s-column*, Nonetheless we will, so to have the verification-row.

```
:: (setq D (mat-det M))  =>  40
:: (with-ring Integer-ring (lightning Prime D))

      n:   r_n   q_n   s_n   t_n
    /--------------------------\
      0:    53    --     1     0
      1:    40     1     0     1
      2:    13     3     1    -1
      3:     1    13    -3     4
      4:     0 Infty    40   -53
    \__________________________/

      Rename r3,s3,t3  to  GCD,S,T.

      Note    GCD  =  r0 * S  +  r1 * T,   i.e
                1  =  [53]*[-3] + [40]*[4].
```

The Upshot: The mod-53 reciprocal of 40 is 4. I.e, $\langle 1/40 \rangle_{53} = 4$. ♦

Computing M^{-1} for fun. Let's "go the extra mile" and otain the multiplicative -nverse of M.

Below, let $\equiv$ mean $\equiv_{53}$. As $\mathrm{Det}(\mathsf{M}) \neq 0$, we have that

$$\mathsf{M}^{-1} \equiv \frac{1}{\mathrm{Det}(\mathsf{M})} \begin{bmatrix} 5 & -3 \\ -4 & 21 \end{bmatrix} \equiv 4{\cdot}\begin{bmatrix} 5 & -3 \\ -4 & 21 \end{bmatrix} \equiv \begin{bmatrix} 20 & -12 \\ -16 & 84 \end{bmatrix} \equiv \begin{bmatrix} 20 & 41 \\ 37 & 31 \end{bmatrix}.$$

We can check correctness by multiplying M by M^{-1} in both orders:

```
:: (setq invM (mat-inverse? M))
    [  20  41  ]
    [  37  31  ]

:: (mat-mul invM M)
    [  1  0  ]
    [  0  1  ]

:: (mat-mul M invM)
    [  1  0  ]
    [  0  1  ]
```

Looks good. . . To see one computation before reducing mod-53:

```
:: (with-ring Integer-ring (mat-mul invM M))
    [  584  265  ]
    [  901  266  ]
```

This latter is mod-53 congruent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Could you solve an analogous problem over $\mathbb{C}$, the complex field?

Q6: Fri. 15 Sep Over $\mathbb{C}$, let $\mathsf{M} := \begin{bmatrix} 1+3\mathbf{i} & 7\mathbf{i} \\ \mathbf{i} & 2\mathbf{i} \end{bmatrix}$. In standard form, $\mathrm{Det}(\mathsf{M}) =$ ______. Its reciprocal is $\frac{1}{\mathrm{Det}(\mathsf{M})} = x + \mathbf{i}y$, for reals $x=$ ______ and $y=$ ______.

The $(1,1)$-entry of M^{-1} is ______ + [______]$\mathbf{i}$.

2×2 matrix determinant.

```
:: (use-ring GaussRational-ring)
:: (setq M (mat-make-square
      '( #C(1 3) #C(0 7) #C(0 1) #C(0 2) )))

   [ 1+3i  7i ]
   [    i  2i ]
```

So $D := \mathrm{Det}(\mathsf{M}) = [[1+3\mathbf{i}]\cdot 2\mathbf{i}] - [7\mathbf{i}\cdot\mathbf{i}] = 1+2\mathbf{i}$.

```
:: (setq D (mat-det M)) => 1+2i
```

Thus, $1/D = \overline{D}/|D|^2 = \dfrac{1-2\mathbf{i}}{5}$.

Hence $x = 1/5$ and $y = \text{-}2/5$.

Examining M^{-1}. Note

$$\mathsf{M}^{-1} = \frac{1}{\mathrm{Det}(\mathsf{M})}\begin{bmatrix} 2\mathbf{i} & -7\mathbf{i} \\ -\mathbf{i} & 1+3\mathbf{i} \end{bmatrix} = \left[\tfrac{1}{5} - \tfrac{2}{5}\mathbf{i}\right]\begin{bmatrix} 2\mathbf{i} & -7\mathbf{i} \\ -\mathbf{i} & 1+3\mathbf{i} \end{bmatrix}.$$

Hence the $(1,1)$-entry of M^{-1} is $\dfrac{4}{5} + \dfrac{2}{5}\mathbf{i}$. ♦

Computing M^{-1} for fun. Multiplying out:

```
:: (setq invM (mat-inverse? M))

   [  [4/5]+[2/5]i     [-14/5]-[7/5]i ]
   [ [-2/5]-[1/5]i       [7/5]+[1/5]i ]
```

A nicer way to write this is

$$\mathsf{M}^{-1} = \frac{1}{5}\cdot\begin{bmatrix} 4+2\mathbf{i} & -14-7\mathbf{i} \\ -2-\mathbf{i} & 7+\mathbf{i} \end{bmatrix}.$$

Could you solve an analogous problem over $\mathbb{Z}_{13}$?

Q7: Mon. 18 Sep *Here*, let *AT* mean "Always True", *AF* mean "Always False" and *Nei* mean "Neither always true nor always false". Below, $\mathbf{v}, \mathbf{w}, \mathbf{x}$ repr. *distinct, non-zero* vectors in $\mathbb{R}^4$, a $\mathbb{R}$-VS. Please $\boxed{\text{circle}}$ the correct response:

y1 If $\mathbf{x} \notin \mathrm{Spn}\{\mathbf{v}, \mathbf{w}\}$ then $\{\mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is linearly independent.
AT *AF* *Nei*

y1. CEX: Let $\mathbf{v} := \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $\mathbf{w} := 2\mathbf{v}$, and $\mathbf{x} := \left[\begin{smallmatrix}7\\0\end{smallmatrix}\right]$. ♦

y2 Collection $\{\mathbf{0}, \mathbf{x}\}$ is linearly-independent.
AT *AF* *Nei*

y2. Sum $1\cdot\mathbf{0} + 0\cdot\mathbf{x} = \mathbf{0}$, yet 1 is not zero in $\mathbb{R}$. ♦

y3 $\mathrm{Spn}\{\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{v}+2\mathbf{w}+3\mathbf{x}\}$ is all of $\mathbb{R}^4$.
AT *AF* *Nei*

y3. The given span equals $\mathcal{S} := \mathrm{Spn}\{\mathbf{v}, \mathbf{w}, \mathbf{x}\}$, since $\mathbf{v}+2\mathbf{w}+3\mathbf{x} \in \mathcal{S}$. But $\mathcal{S}$ is at most 3-dimensional. ♦

y4 If none of $\mathbf{v}, \mathbf{w}, \mathbf{x}$ is a multiple of the other vectors, then $\{\mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is linearly independent.
AT *AF* *Nei*

y4. CEX: Let $\mathbf{v} := \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $\mathbf{w} := \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathbf{x} := \mathbf{v}+\mathbf{w} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. ♦

y5 For 2×2 matrices: $\mathrm{Det}(\mathsf{B}+\mathsf{A}) = \mathrm{Det}(\mathsf{B}) + \mathrm{Det}(\mathsf{A})$.
AT *AF* *Nei*

y5. CEX: Matrices $\mathsf{B} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathsf{A} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ each have determinant zero, yet $\mathsf{B}+\mathsf{A}$ is the identity-matrix, which has $\mathrm{Det} = 1$.

OTOHand, $\mathrm{Det}(\mathsf{B}+\mathsf{B})$ does equal $\mathrm{Det}(\mathsf{B}) + \mathrm{Det}(\mathsf{B})$. ♦

Q8: Tue. 19 Sep Matrix-product $\begin{bmatrix} b \\ c \end{bmatrix} \cdot \begin{bmatrix} x & y \end{bmatrix} =$ ______________ .

Soln. We have a $2{\times}1$ times a $1{\times}2$, hence the output is a $2{\times}2$ matrix; $\begin{bmatrix} bx & by \\ cx & cy \end{bmatrix}$. ♦

Monday, 25Sep. is Yom Kippur, no MQ; if you miss class, get notes from several colleagues. Aaron Thrasher will teach.

Continued on next page...

Q9: Tue. 26 Sep $\mathsf{M} := \begin{bmatrix} 70 & 7 \\ 1 & 2 \end{bmatrix}$. Compute M^{-1} over these three fields. [Write your $\mathbb{Z}_p$ answers using *symmetric* residues.]

Over $\mathbb{Z}_{13}$: $\mathsf{M}^{-1} =$ ______. Over $\mathbb{Z}_7$: $\mathsf{M}^{-1} =$ ______

Over $\mathbb{Q}$: $\mathsf{M}^{-1} =$ ______________________.

Soln: Mat-inverse over $\mathbb{Z}_{13}$.

```
% (setq M (mat-make-square 70 7 1 2))
    [ 70   7  ]
    [  1   2  ]

% (use-ring Modsym-M-ring) ;; Set the ring.
% (set-ring-modulus 13)
```

Reducing, $\mathsf{M} \equiv_{13} \begin{bmatrix} 5 & -6 \\ 1 & 2 \end{bmatrix}$. *Consequently,*

```
% (mat-inverse? M)
    [  5   2  ]
    [  4   6  ] .
```

By hand: $\mathrm{Det}(\mathsf{M}) = 10 + 6 \equiv_{13} 3$. And $\langle 1 \div 3 \rangle_{13} \equiv -4$. Thus $\mathsf{M}^{-1} \equiv (-4) \cdot \begin{bmatrix} 2 & 6 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -8 & -24 \\ 4 & -20 \end{bmatrix} \equiv \begin{bmatrix} 5 & 2 \\ 4 & 6 \end{bmatrix}$. ♦

Or: If you prefer, $\langle 1 \div 3 \rangle_{13} \equiv 9$, whence

$$\mathsf{M}^{-1} \equiv 9 \cdot \begin{bmatrix} 2 & 6 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 54 \\ -9 & 45 \end{bmatrix} \equiv \begin{bmatrix} 5 & 2 \\ 4 & 6 \end{bmatrix}.$$

To Be or not to Be; that is the mod-7 question.

```
% (set-ring-modulus 7) ;; Change the modulus.
;; Well, M is mod-7 congruent to
    [ 0   0  ]
    [ 1   2  ] ,
;; which has zero-determinant, hence is not
;; invertible.   Or, we can run the code...

% (mat-inverse? M)
  JK: Found 1 pivots before column 2.
NIL
```

In Lisp-speak, NIL means "DNE", "false". I.e, M has no mod-7 inverse. ♦

Rationality at last... Computing by hand, $\mathrm{Det}(\mathsf{M}) = [70 \cdot 2] - [7 \cdot 1] = 133$, *etc.* By machine,

```
% (use-ring Rational-ring)   ;; Change the ring.

% (mat-inverse? M)
  JK: Found 2 pivots before column 2.
[  2/133   -1/19  ]
[ -1/133   10/19  ]
```

I.e $\mathsf{M}^{-1} = \begin{bmatrix} 2/133 & -1/19 \\ -1/133 & 10/19 \end{bmatrix} \overset{\text{note}}{=\!=} \frac{1}{133} \cdot \begin{bmatrix} 2 & -7 \\ -1 & 70 \end{bmatrix}$, noting that $133 = 7 \cdot 19$. ♦

Wedn., 27Sep. is Class R, our first in-class exam, which will have a Change-of-Basis question.

This is a closed-book, open-brain exam. Bring both lined and unlined paper. Bring several writing implements. Make sure they write **Dark**.

No faint pencil !

QA: Tue. 03 Oct Let $\mathbf{V} := \mathsf{P}_2(\mathbb{R})$, the VS of 3-topped $\mathbb{R}$-polynomials. Inside VS $\mathrm{MAT}_{2\times 2}(\mathbb{R})$, consider the VSS, $\mathbf{U}$, of upper-triangular matrices. Prove

$$\mathbf{U} \overset{\text{VS-iso}}{\cong} \mathbf{V};$$

i.e, that $\mathbf{U}$ and $\mathbf{V}$ are vector-space isomorphic.

Addendum for Posting: [Not part of the MQ.] Consider linear map $\mathsf{T}{:}\mathbf{V}\circlearrowleft$ by

$$\mathsf{T}(h) := h + h' + h(3).$$

Having constructed a particular iso $\Phi{:}\mathbf{V} \hookrightarrow\!\!\!\!\!\rightarrow \mathbf{U}$, use it to carry T over to $\mathbf{U}$. That is, give an explicit description of the action of $\boxed{\mathsf{S} := \Phi \circ \mathsf{T} \circ \Phi^{-1}}$ on the space of upper-triangular matrices.

QB: Wed. 11 Oct With $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the standard basis for $\mathbb{R}^3$, and $\{\mathbf{v}_1, \mathbf{v}_2\}$ the std basis for $\mathbb{R}^2$, define linear map $\mathsf{S}{:}\mathbb{R}^3 \to \mathbb{R}^2$ by: $\mathsf{S}(\mathbf{e}_1) = 17\mathbf{v}_1 - 2\mathbf{v}_2$ and $\mathsf{S}(\mathbf{e}_2) = 6\mathbf{v}_2$ and $\mathsf{S}(\mathbf{e}_3) = \text{-}4\mathbf{v}_1 - 3\mathbf{v}_2$.

The *matrix* of the lefthand-action of S is: ....

Soln. This matrix, M, acts on colvecs of height 3, producing height-2 colvecs. Hence M is 2×3. Indeed

$$\mathsf{M} = \begin{bmatrix} 17 & 0 & \text{-}4 \\ \text{-}2 & 6 & \text{-}3 \end{bmatrix}. \quad ♦$$

QC: Fri. 13 Oct Over field $\mathbb{Z}_7$, consider matrix

$$\mathsf{A} := \begin{bmatrix} 16 & 10 & -8 \\ 3 & 37 & 12 \\ -33 & 27 & 23 \end{bmatrix}.$$

Then

$$\mathsf{A}^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}.$$

Write entries in $[\text{-}3\,..\,3]$. Also, $\mathrm{Det}(\mathsf{A})=$ ....

Helpful tables:

Modulo 7:

x	$\langle 1/x\rangle_7$	x	$\langle 1/x\rangle_7$
± 1	± 1		
± 2	∓ 3	± 3	∓ 2

Multiplication mod 7:

```
 7 |  2   3
----|--------
 2 | -3
 3 | -1   2
```

Soln. First, let's input the matrix, then reduce it mod 7.

```
(progn (set-ring-modulus 7)(use-ring Modsym-M-ring))
(setq AOrig (mat-make-square   16  10  -8
                                3  37  12
                              -33  27  23 ))

;; Make the code  reduce mod-7, by multiplying
;; by the identity matrix.
(setq A (mat-mul (mat-eye 3) AOrig))
    [  2   3  -1  ]
    [  3   2  -2  ]
    [  2  -1   2  ]


;; Adjoin the identity, to make a 3x6 matrix,
;; then RREF...
(dsetq (DetA InvA) (mat-det-inverse AOrig))

(progn DetA)  =>  2

(progn InvA)
    [  1   1  -2  ]
    [  2   3  -3  ]
    [  0  -3   1  ]

;;Verify.
(mat-mul A InvA)
    [  1  0  0  ]
    [  0  1  0  ]
    [  0  0  1  ]
```

♦

QD: Mon. 16 Oct Over $\mathbb{Q}$, determinant $\mathrm{Det}(\mathsf{D})=$ ____________
where
$$\mathsf{D} := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ -2 & 1 & 1 \end{bmatrix}.$$

Soln. We *could* use the Rule of Sarrus, but let's instead use RREF and compute the $\mathbb{Q}$-inverse of D, as well as D's determinant.

```
(setq D (mat-make-square
    1    2    3
    0    5    4
   -2    1    1 ))

(dsetq (DetD  InvD) (mat-det-inverse D))

DetD  =>  15

InvD  =>
    [  1/15   1/15   -7/15 ]
    [ -8/15   7/15   -4/15 ]
    [  2/3    -1/3    1/3  ]

(mat-mul InvD D)     =>
    [  1   0   0  ]
    [  0   1   0  ]
    [  0   0   1  ]
```

♦

QE: Fri. 20 Oct

Perm $\pi := [6, 7, 8, 1, 2, 3, 4, 5]$ has $\mathrm{Sgn}(\pi) =$ +**1** $\quad$ **−1**.

Perm-sign: This π switches a 3-block with a 5-block, hence can be realized with $3 \cdot 5 = 15$ transpositions, hence is an odd perm.

Alternatively, since $3 \perp 8$, this block-switch makes π a single 8-cycle, so the number of even-len-cycles [***EL-cycles***] is 1. Thus $\mathrm{Sgn}(\pi) = [-1]^1 = -1$.

In CCN [Canonical Cycle Notation],

$$\pi = (8 \to 5 \to 2 \to 7 \to 4 \to 1 \to 6 \to 3).$$

QF: Mon. 23 Oct Let $\mathsf{M} := \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ -3 & 1 & 0 \end{bmatrix}$ and $\mathsf{T} := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Mystery vector $\mathsf{X} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies $\mathsf{MX} = \mathsf{T}$. Compute $\mathrm{Det}(\mathsf{M}_{2,\mathsf{T}})$, where $\mathsf{M}_{2,\mathsf{T}}$ is M with its 2nd-column replaced by T. Compute $\mathrm{Det}(\mathsf{M})$. Optionally, compute x_2.

Cramer's thm. Recall that Lisp counts from zero, so our row/col numbers in Lisp code are 0,1,2.

```
% (use-ring Rational-ring)

% (setq M (mat-make-square 1 2 4  0 2 1  -3 1 0))
    [   1  2  4  ]
    [   0  2  1  ]
    [  -3  1  0  ]

;; The Target column-vector:
% (setq Tar (mat-make-colvec 1 2 3))
    [  1  ]
    [  2  ]
    [  3  ]

;; Substituting Tar into the middle-column.
% (setq M1Tar (mat-set-submatrix (copy-mat M) 0 1 Tar))
    [   1  1  4  ]
    [   0  2  1  ]
    [  -3  3  0  ]

;; Numerator determinant:
% (mat-det M1Tar)  =>  18

;; Denominator determinant:
% (mat-det M)      =>  17

;; Hence the second entry of colvec X is 18/17.

;;;; Let's get all of X:
% (setq InvM (mat-inverse? M))
    [  -1/17   4/17  -6/17  ]
    [  -3/17  12/17  -1/17  ]
    [   6/17  -7/17   2/17  ]

% (setq XAns (mat-mul InvM Tar))
    [  -11/17  ]
    [   18/17  ]
    [   -2/17  ]
```

♦

Exercise: Re-solve this, viewing M and T as matrices over field $\mathbb{Z}_{13}$.

Continued on next page...

QG: Tue. 24 Oct Let $\mathsf{M}(x) := \begin{bmatrix} x+1 & 2x & 1 \\ -1 & 1 & 1 \\ 0 & x-1 & 0 \end{bmatrix}$. Then

$\mathrm{Det}\big(\mathsf{M}(x)\big) = \underline{\;2\;} + \underline{\;-1\;}\, x + \underline{\;-1\;}\, x^2 + \underline{\;0\;}\, x^3.$

Soln. Using the Maple software:

```
M := <<x+1, -1, 0>|<2*x, 1, x-1>|<1,1,0>>;

                [x + 1      2 x      1]
                [                     ]
           M := [ -1         1       1]
                [                     ]
                [  0       x - 1     0]



                                2
Det(M)  =>                   -x   + 2 - x
```

♦

Wednesday, 25Oct2023 is CLASS S, our second in-class exam.

This is a closed-book, open-brain exam. Bring both lined and unlined paper. Bring <u>several</u> writing implements. Make sure they write (defun h (a b c d e f) (setq Id-ToE-FrB (mat-Horiz-concat eVec4 (mmc a b c)(mmc d e f)) NewM (mat-mul (mat-inverse? Id-ToE-FrB) M Id-ToE-FrB))) (defun h (a b c d e f) (setq Id-ToE-FrB (mat-Horiz-concat eVec4 (mmc a b c)(mmc d e f)) NewM (mat-mul (mat-inverse? Id-ToE-FrB) M Id-ToE-FrB)))

Dark.

No *faint pencil* ***!***

QH: Mon. 30 Oct Working over $\mathbb{Z}_{11}$, let $\mathsf{M} := \begin{bmatrix} -2 & 0 & 4 \\ 1 & 2 & 3 \\ 98 & 3 & 1 \end{bmatrix}$. Then characteristic-poly of M

is $\wp_{\mathsf{M}}(t) = -t^3 + t^2 - 2t + 1$.

[Write coeffs as symmetric-residues, in [-5 .. 5].]

Soln. Reducing, $\begin{bmatrix} -2 & 0 & 4 \\ 1 & 2 & 3 \\ 98 & 3 & 1 \end{bmatrix} \equiv_{11} \begin{bmatrix} -2 & 0 & 4 \\ 1 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} =: \mathsf{M}$

Char-poly is $\mathrm{Det}(\mathsf{M} - t\mathbf{I}) = \mathrm{Det}\left(\begin{bmatrix} -2-t & 0 & 4 \\ 1 & 2-t & 3 \\ -1 & 3 & 1-t \end{bmatrix}\right)$ which, by cofactor along the top row, equals

$$[-2-t]\cdot\mathrm{Det}\left(\begin{bmatrix} 2-t & 3 \\ 3 & 1-t \end{bmatrix}\right) - 0 + 4\cdot\mathrm{Det}\left(\begin{bmatrix} 1 & 2-t \\ -1 & 3 \end{bmatrix}\right).$$

Let's compute:

```
% (setq *POLY-var*  "t")
% (progn (set-ring-modulus 11)(use-ring Modsym-M-ring))
% (setq MOrig (mat-make-square    -2   0   4
                                   1   2   3
                                  98   3   1 ))

;; Reduce mod-11, via multiplying by the identity matrix.
% (setq M (mat-mul (mat-eye 3) MOrig))
     [ -2  0  4 ]
     [  1  2  3 ]
     [ -1  3  1 ]

% (mat-CharMat M)
     [ -t-2     0      4 ]
     [    1  -t+2      3 ]
     [   -1     3   -t+1 ]

% (mat-CharPoly MOrig)       => -t^3 + t^2 - 2t + 1
% (setq pM (mat-CharPoly M)) => -t^3 + t^2 - 2t + 1

% (iter (for j :from -5 :to 5)
        (format t "~%p(~2D) = ~2D" j (poly-eval pM j)) )

   p(-5) = -4    p(-2) = -5    p( 1) = -1    p( 4) =  0
   p(-4) =  1    p(-1) =  5    p( 2) =  4    p( 5) =  1
   p(-3) = -1    p( 0) =  1    p( 3) = -1
```

As 4 is zero of $\wp_{\mathsf{M}}$, `(poly-divide pM (poly-cree -1 4))` yields $\boxed{\wp_{\mathsf{M}}(t) = -1\cdot[t-4]\cdot[t^2+3t+3]}$.

Since $\mathrm{Char}(\mathbb{Z}_{11}) \neq 2$, we can apply the quadratic formula to $f(t) := t^2 + 3t + 3$. We already know f is $\mathbb{Z}_{11}$-irreducible, so $\mathrm{Discr}(f) \neq \square$, but let's check anyway: $\mathrm{Discr}(f) = 3^2 - 4\cdot 3 \equiv -3$.

Yet the mod-11 squares are $1, 4, -2, 5, 3$.

Getting fancy. In grad-school algebra, one learns that there is an extension field $\mathbb{G} \supset \mathbb{Z}_{11}$ over which f factors completely. [The smallest such is called the "***splitting field*** of f over $\mathbb{Z}_{11}$".] Over $\mathbb{G}$, then, our char-poly factors completely as

$$\wp_{\mathsf{M}}(t) = -1\cdot[t-4]\cdot[t-\boldsymbol{\alpha}]\cdot[t-\boldsymbol{\beta}],$$

for some some $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{G} \setminus \mathbb{Z}_{11}$.

If $\boldsymbol{\alpha}=\boldsymbol{\beta}$, then M *might* be diagonalizable. However, if $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$, then M *will be* $\mathbb{G}$-conjugate to $\begin{bmatrix} 4 & 0 & 0 \\ 0 & \boldsymbol{\alpha} & 0 \\ 0 & 0 & \boldsymbol{\beta} \end{bmatrix}$. So the question becomes...

Does f have a repeated root? In our current setting, f has a repeated root IFF $\mathrm{GCD}(f, f')$ is non-trivial.

In our case, $f'(t) = 2t+3$. Since f' is degree-1, were $\mathrm{GCD}(f, f')$ non-trivial, then $\boxed{f' \text{ would divide } f}$. But f' is degree-1, so f' has a root in $\mathbb{Z}_{11}$; yet we know that f does not. THE UPSHOT:

Matrix M is not diagonalizable over $\mathbb{Z}_{11}$, but is[♡1] *$\mathbb{G}$-diagonalizable, with three distinct eigenvalues.*

A better basis? A basis for $\mathrm{LNul}(\mathsf{M} - 4\mathbf{I})$ will have a single vector.

```
% (setq M-4I  (mat-sub M (mat-scal-mult 4 (mat-eye 3))))
    [  5   0   4 ]
    [  1  -2   3 ]
    [ -1   3  -3 ]
                                             [ -3 ]
% (setq eVec4 (mat-LNul M-4I))   =>          [  0 ]
                                             [  1 ]
```

To this eVec, let's automate expanding it to a basis, then converting M to that basis.

```
% (defaliasqq mmc mat-make-colvec)
% (defun CoB-automator (a b c d e f)
   (setq
     Id-ToE-FrB (mat-Horiz-concat eVec4 (mmc a b c)(mmc d e f))
     NewM       (mat-mul (mat-inverse? Id-ToE-FrB) M Id-ToE-FrB)
     ) )
                                         [ 4   0   1 ]
% (CoB-automator 0 1 0 0 0 2)  =>        [ 0   2  -5 ]
                                         [ 0  -4  -5 ]
```

Challenge:

Is M similar to a matrix OTForm $\begin{bmatrix} 4 & & \\ & a & b \\ & c & d \end{bmatrix}$??

[♡1] This is analagous to the following:
The general rotation matrix is not diagonalizable over $\mathbb{R}$, but is diagonalizable over $\mathbb{C}$.

Ungiven MQ: Fri. 03 Nov Over $\mathbb{C}$, matrix $G := \begin{bmatrix} -1 & 0 & 1 \\ 0 & \mathbf{i} & 0 \\ -2 & 0 & 2 \end{bmatrix}$ is diagonalizable. Char-poly $\wp_G(t) =$ ______________, and factors as $\wp_G(t) = -1 \cdot [t - \_\_\_] \cdot [t - \_\_\_] \cdot [t - \_\_\_]$.

Invertible matrix $U =$ ______________________ has that $D := U^{-1}GU$ is diagonal, where $D =$ ________________.

Soln. Our CoB matrix will be $U := [\mathbf{b}_0\ \mathbf{b}_1\ \mathbf{b}_2]$ for G-eVecs $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ that we will compute.

The middle column of G shows $\mathbf{i}$ is an eVal, with eVec $\mathbf{b}_0 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Cols 0 and 2 of G add to the zero-vector, so G has nt-nullspace. We can let $\mathbf{b}_1 := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ be an eVec for eVal 0.

We now know $G \overset{\text{sim}}{\sim} \begin{bmatrix} \mathbf{i} & & ? \\ & 0 & ? \\ & & ? \end{bmatrix}$; actually $G \overset{\text{sim}}{\sim} \begin{bmatrix} \mathbf{i} & & \\ & 0 & \\ & & ? \end{bmatrix}$; since we were assured G is diagonalizable.

Char-poly. Recall $\wp_G(t) = -t^3 + \operatorname{Tr}(G)t^2 + ?t + \operatorname{Det}(G)$. So $\boxed{\wp_G(t) = -t^3 + [1+\mathbf{i}]t^2 + ?t}$. *Sigh. We can't put it off any longer; let's compute:*

```
(setq *POLY-var* "t") (use-ring GaussRational-ring)
(setq G (mat-make-square    -1   0   1
                             0  imi  0
                            -2   0   2  ))

(setq p (mat-CharPoly G)) => -t^3 + (1+i)t^2 + (-i)t
(setq t-0 (poly-cree 1 0)    t-i (poly-cree 1 (- imi)))

(dsetq (pDiv-t0 Rem) (poly-divide p t-0))
   -t^2 + (1+i)t + (-i)                    <ZIP poly>

(dsetq (pDiv-t0-ti Rem) (poly-divide pDiv-t0 t-i))
   -t+1                                    <ZIP poly>
```

Cool. A smidgeon of elbow-grease has shown that

$$\wp_G(t) = -1 \cdot [t - \mathbf{i}] \cdot [t - 0] \cdot [t - 1].$$

So . . . ♦

Computing an eVec for eVal=1.

```
(setq G-1I (mat-sub G (mat-eye 3)))
    [ -2     0  1 ]
    [  0  -1+i  0 ]
    [ -2     0  1 ]
                       [ 1/2 ]
(mat-LNul G-1I)    =>  [  0  ]
                       [  1  ]
                                              [ 1 ]
(setq b2 (mat-make-colvec 1 0 2))  =>  [ 0 ]
                                              [ 2 ]

(setq U (mat-Horiz-concat b0 b1 b2))
    [ 0  1  1 ]
    [ 1  0  0 ]
    [ 0  1  2 ]

(setq InvU (mat-inverse? U))
    [  0  1   0 ]
    [  2  0  -1 ]
    [ -1  0   1 ]

(setq CDiagonalized (mat-mul InvU G U))
    [ i  0  0 ]
    [ 0  0  0 ]
    [ 0  0  1 ]
```

Nifty. ♦

Ungiven MQ: Mon. 06 Nov P.279#7 Over $\mathbb{C}$, let $\mathsf{A} := \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
For $n = 1, 2, 3, \ldots$, $\mathsf{A}^n =$ ______________________________.

Soln. The idea is to see if A is conjugate to a diagonal matrix.

```
(setq A (mat-make-square 1 4 2 3))  =>  [ 1  4 ]
                                        [ 2  3 ]

(setq pA (mat-CharPoly A))  =>  t^2 - 4t - 5

;; So eVals are 5 and -1.
(setq I2 (mat-eye 2))  =>  [ 1  0 ]
                           [ 0  1 ]

(setq b0 (mat-LNul (mat-sub A (mat-scal-mult  5 I2)))
      b1 (mat-LNul (mat-sub A (mat-scal-mult -1 I2))) )
        [ 1 ]    [ -2 ]
        [ 1 ] ,  [  1 ]

(setq U (mat-Horiz-concat b0 b1)  InvU (mat-inverse? U))
        [ 1  -2 ]    [  1/3  2/3 ]
        [ 1   1 ] ,  [ -1/3  1/3 ]

;;Checking...
(setq D (mat-mul InvU A U)) => [ 5   0 ]
                               [ 0  -1 ]
```

Consequently

$$\mathsf{A}^n = \mathsf{U}\mathsf{D}^n\mathsf{U}^{-1} = \frac{1}{3}\begin{bmatrix} 5^n + 2\cdot[-1]^n & 2\big(5^n - [-1]^n\big) \\ 5^n - [-1]^n & 2\cdot 5^n + [-1]^n \end{bmatrix}.$$

Elegant... ♦

Testing our formula.

```
(defun Formula-A^n (n)
  (setq F (expt  5 n) ;; Five
        G (expt -1 n) ;; neG-one
   )
  (mat-scal-mult 1/3
    (mat-make-square (+ F G G)   (* 2 (- F G))
                     (- F G)     (+ F F G))
     ) )


(defun Test-A^n (Bnd)
  (iter (for n :from 0 :below Bnd)
    (for A^n :first (mat-eye 2) :then (mat-mul A^n A))
    (format t "~%~2D: ~A. ~A" n (Formula-A^n  n) A^n)
    ) )

;;Testing the formula...
(Test-A^n 6)
     0: [ 1  0 ]   [ 1  0 ]
        [ 0  1 ] , [ 0  1 ]

     1: [ 1  4 ]   [ 1  4 ]
        [ 2  3 ] , [ 2  3 ]

     2: [ 9  16 ]   [ 9  16 ]
        [ 8  17 ] , [ 8  17 ]

     3: [ 41  84 ]   [ 41  84 ]
        [ 42  83 ] , [ 42  83 ]

     4: [ 209  416 ]   [ 209  416 ]
        [ 208  417 ] , [ 208  417 ]

     5: [ 1041  2084 ]   [ 1041  2084 ]
        [ 1042  2083 ] , [ 1042  2083 ]
```

♦

QI: Mon. 20 Nov In $\mathbb{R}^3$, consider colvecs $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}0\\0\\5\end{bmatrix},$$

respectively. With $\mathrm{IP} = \mathrm{DotProduct}$, compute pairwise orthogonal $\mathbf{b}_1 := \mathbf{g}_1, \mathbf{b}_2, \mathbf{b}_3$ using Gram-Schmidt, yielding

$$\mathbf{b}_2 = [\ -1\ ,\ 1\ ,\ 1\ ]^{\mathrm{t}} \quad \textit{and}$$

$$\mathbf{b}_3 = [\ -5/6\ ,\ -5/3\ ,\ 5/6\ ]^{\mathrm{t}}.$$

Soln. Using our formula for ortho-projection:

```
(ip-select t)  =>  "Using COLVEC-inner-product."
(use-ring GaussRational-ring)
(defaliasqq mmc mat-make-colvec)
(setq g1 (mmc 1    0     1  )
      g2 (mmc 2    1     4  )
      g3 (mmc 0    0     5  ) )

(setq B1 G1) ;; Computing B1:

;; Computing B2:
(setq ratio (/ (ip B1 G2) (ip B1 B1))) => 3
(setq ProjG2onB1 (mat-scal-mult ratio B1)) => [ 3 ]
                                              [ 0 ]
                                              [ 3 ]

(setq B2 (mat-sub G2 ProjG2onB1))  =>  [ -1 ]
                                       [  1 ]
                                       [  1 ]

(setq ratio (/ (ip B1 G3) (ip B1 B1))) => 5/2
(setq ProjG3onB1 (mat-scal-mult ratio B1)) =>  [  5/2 ]
                                               [   0  ]
                                               [  5/2 ]

(setq ratio (/ (ip B2 G3) (ip B2 B2))) => 5/3
(setq ProjG3onB2 (mat-scal-mult ratio B2)) =>  [ -5/3 ]
                                               [  5/3 ]
                                               [  5/3 ]

(setq B3 (mat-sub G3 (mat-add ProjG3onB1 ProjG3onB2)))
    [ -5/6 ]
    [ -5/3 ]
    [  5/6 ]

;; Checking pairwise orthogonality:
(list (ip b1 b2) (ip b1 b3) (ip b2 b3))  =>  (0 0 0)
```

♦

QJ: Tue. 21 Nov In $\mathbb{C}^4$, consider $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$, column-vectors

$$\begin{bmatrix}0\\ \mathbf{i}\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 0\\ 2\mathbf{i}\end{bmatrix}, \begin{bmatrix}1\\ 0\\ 1+\mathbf{i}\\ 1\end{bmatrix},$$

respectively. With IP = DotProduct, compute pairwise orthogonal $\mathbf{b}_1 := \mathbf{g}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ using Gram-Schmidt, yielding

$$\mathbf{b}_2 = [\ 0\ ,\ 0\ ,\ 0\ ,\ 2\mathbf{i}\ ]^{\mathrm{t}} \text{ and}$$

$$\mathbf{b}_3 = [\ 1\ ,\ 0\ ,\ 1+\mathbf{i}\ ,\ 0\ ]^{\mathrm{t}}.$$

Soln. This triple can be orthogonalized "by inspection". Alternatively,. . .

```
(ip-select t)  =>  "Using COLVEC-inner-product."
(use-ring GaussRational-ring)
(defaliasqq mmc mat-make-colvec)
(setq g1 (mmc 0    imi    0       0      )
      g2 (mmc 0      1    0       #C(0 2))
      g3 (mmc 1      0    #C(1 1) 1      ) )
  [ 0 ]    [  0 ]   [   1 ]
  [ i ]    [  1 ]   [   0 ]
  [ 0 ] ,  [  0 ] , [ 1+i ]
  [ 0 ]    [ 2i ]   [   1 ]

(setq B1 G1) ;; "Computing" B1.

;; Computing B2:
(setq ratio (/ (ip B1 G2) (ip B1 B1))) => -i    [ 0 ]
(setq ProjG2onB1 (mat-scal-mult ratio B1)) =>   [ 1 ]
                                                [ 0 ]
                                                [ 0 ]
                                          [  0 ]
(setq B2 (mat-sub G2 ProjG2onB1))  =>     [  0 ]
                                          [  0 ]
                                          [ 2i ]

(setq ratio (/ (ip B1 G3) (ip B1 B1))) => 0
;; Already perpendicular, cool.


(setq ratio (/ (ip B2 G3) (ip B2 B2)))  => -[1/2]i

(setq ProjG3onB2 (mat-scal-mult ratio B2)) =>   [ 0 ]
                                                [ 0 ]
                                                [ 0 ]
                                     [   1 ]    [ 1 ]
(setq B3 (mat-sub G3 ProjG3onB2)) => [   0 ]
                                     [ 1+i ]
                                     [   0 ]
```

♦

Ans. We test whether the **b**-tuple is pairwise-ortho:

```
(list (ip b1 b2) (ip b1 b3) (ip b2 b3))  =>  (0 0 0)


;;;;;;;;;;;;;;;;
;; Checking that Span(B1, B2, B3) = Span(G1, G2, G3).
;;;;;;;;;;;;;;;;
(setq Bert (mat-Horiz-concat B1 B2 B3   G1 G2 G3))
   [ 0   0    1    0   0    1 ]
   [ i   0    0    i   1    0 ]
   [ 0   0  1+i    0   0  1+i ]
   [ 0  2i    0    0  2i    1 ]


(rref-mtab-beforecol Bert)
  JK: Found 3 pivots before the sixth column.

 c0 c1 c2   c3 c4  c5          Row operations
|------------------------|
| 1  0  0    1 -i    0     | 0  -i    0              0
| 0  1  0    0  1 -[1/2]i | 0   0    0          -[1/2]i
| 0  0  1    0  0    1     | 0   0  [1/2]-[1/2]i     0
| 0  0  0    0  0    0     | 1   0  [-1/2]+[1/2]i    0
|------------------------|
```

The *RREF* shows that, were we to exchange columns c_4, c_5, c_6 with c_1, c_2, c_3, we would have three pivits in c_4, c_5, c_6. Consequently $\mathrm{Spn}(c_4, c_5, c_6) = \mathrm{Spn}(c_1, c_2, c_3)$. ♦

QK: Mon. 27 Nov The real-IPS $\mathsf{P}(\mathbb{R})$, of polynomials, has inner-product

$$\langle g, h \rangle := \int_0^1 [g \cdot h] .$$

Define $F(x) := x$ and $S(x) := x^2$ [First and Second powers]. Then $\mathrm{Proj}_F(S) = \alpha{\cdot}F$ for scalar $\alpha=$ ____________________.

And $[\mathrm{Orth}_F(S)](x) =$ ____________________.

Channeling my inner product. For non-negative reals J and K, note

$$\left\langle x^J, x^K \right\rangle \stackrel{\text{def}}{=} \int_0^1 x^{J+K}\,\mathrm{d}x = \frac{1}{J+K+1} .$$

Hence

$$\alpha = \frac{\langle x, x^2 \rangle}{\langle x, x \rangle} = \frac{1/[1+2+1]}{1/[1+1+1]} = \frac{3}{4} .$$

Thus,

$$\left[\mathrm{Orth}_F\big(S\big)\right](x) = x^2 - \tfrac{3}{4}\,x . \qquad \blacklozenge$$

QL: Tue. 28 Nov Linear operators $\mathsf{S}, \mathsf{T}, \mathsf{U}$ map $\mathbf{V} \to \mathbf{V}$, where $\mathbf{V}$ is a finite-dim'al $\mathbb{C}$-VS. For $\alpha \in \mathbb{C}$, give a formula for the adjoint operator $[\alpha\mathsf{S} + \mathsf{TU}]^*$.

Soln. Laconically, $[\alpha\mathsf{S} + \mathsf{TU}]^*$ equals

$$[\alpha\mathsf{S}]^* + [\mathsf{TU}]^* = \overline{\alpha}\mathsf{S}^* + \mathsf{U}^*\mathsf{T}^* . \qquad \blacklozenge$$

QM: Tues. 05 Dec Katie's name is $\boxed{\text{circle}}$:

Xavier Aahan Luke Dawson Brighten Yunus Pietro *Katie* Jake Ben Michael George Yun Allan

Games Party: !! Wed. 06 Dec *Bring games and look photogenic, for our traditional* Games Party, from 10:40 am to 4:40 pm, *at* *Pascal's Cafe.*