Sign of a permutation

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12 May, 2024 (at *13:17*)

Below, Ω is a finite set, $N := |\Omega|$, with identity-perm $Id = Id_\Omega$. Use $\mathbb{S} := \mathbb{S}_\Omega$ for the symmetric group $(\mathbb{S}_\Omega, \triangleleft, Id)$. Below β,α are elements of Ω.

Permutation sign. In the cycle-decomposition of perm α, let EL_α be the number of Even-length cycles, and OL_α the number of Odd-length cycles, The ***sign*** of permutation α is

†: $$\mathrm{Sgn}(\alpha) := [\text{-}1]^{\mathrm{EL}_\alpha} .$$

Defn. Perm α is an ***even***/***odd*** permutation as EL_α is even/odd, i.e, as $\mathrm{Sgn}(\alpha)$ is $+1$/-1.

A ***transposition*** is a perm comprising a 2-cycle; the remaining tokens are fixed-pts, i.e, 1-cycles. □

1: Lemma. *Each permutation of a finite set can be written as a composition of transpositions.* ◊

Ways of computation. Let β be this permutation of 7 objects:

□	♣	△	¥	◯	⋇	♡
◯	¥	♡	□	♣	△	⋇

The β cycle-structure is (□ ◯ ♣ ¥)(△ ♡ ⋇); OL_β=one 3-cycle, *and* EL_β=one 4-cycle. So

$$\mathrm{Sgn}(\beta) = [\text{-}1]^{\mathrm{EL}_\beta} = \text{-}1 ;$$

so β is an odd permutation.

Counting transpositions. One transposition sequence is:

□	♣	△	¥	◯	⋇	♡
◯	¥	♡	□	♣	△	⋇
□			◯			
	♣			¥		
		△			♡	
			¥	◯		
					⋇	♡

Five transpositions; so $\mathrm{Sgn}(\beta) = [\text{-}1]^5 = \text{-}1$.

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Counting inversions. If we enumerate □=1, ♣=2, △=3, ◯=7, ¥=6, ♡=5, ⋇=4. Now our β permutation looks like

1	2	3	4	5	6	7
7	6	5	3	4	1	2

The inversions are

$$(1,2), \ldots (1,7),\ (2,3), \ldots (2,7),\ (3,4), \ldots (3,7),$$
$$(4,6), (4,7),\ (5,6), (5,7) .$$

So the inversion number is $6+5+4+2+2 = 19$.

Let's try a different numbering: ◯=2, ¥=4, ♡=6, □=1, ♣=3 △=5,, ⋇=7,

Our β now presents as

1	2	3	4	5	6	7
2	3	4	1	6	7	5

With this enumeration, the inversions are

$$(1,4), (2,4), (3,4), (5,7), (6,7).$$

so the inversion number is 5.