

Sign of a permutation

Jonathan L.F. King
 University of Florida, Gainesville FL 32611-2082, USA
 squash@ufl.edu
 Webpage <http://squash.1gainesville.com/>

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Below, Ω is a finite set, $N := |\Omega|$, with identity perm $Id = Id_\Omega$. Use $\mathbb{S} := \mathbb{S}_\Omega$ for the symmetric group $(\mathbb{S}_\Omega, \triangleleft, Id)$. Below β, α are elements of Ω .

Permutation sign. In the cycle-decomposition of perm α , let EL_α be the number of Even-length cycles, and OL_α the number of Odd-length cycles. The *sign* of permutation α is

$$\dagger: \quad \text{Sgn}(\alpha) := [-1]^{EL_\alpha}.$$

Defn. Perm α is an *even/odd* permutation as EL_α is *even/odd*, i.e., as $\text{Sgn}(\alpha)$ is *+1/-1*.

A *transposition* is a perm comprising a 2-cycle; the remaining tokens are fixed-pts, i.e., 1-cycles. \square

1: Lemma. *Each permutation of a finite set can be written as a composition of transpositions.* \diamond

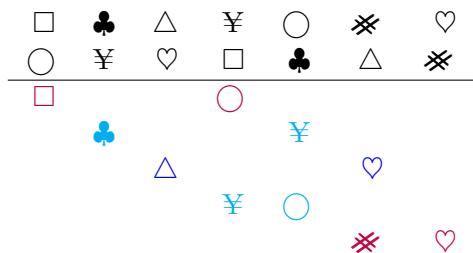
Ways of computation. Let β be this permutation of 7 objects: $\begin{array}{ccccccc} \square & \clubsuit & \triangle & \text{¥} & \circlearrowleft & \divideontimes & \heartsuit \\ \circlearrowright & \text{¥} & \heartsuit & \square & \clubsuit & \triangle & \divideontimes \end{array}$

The β cycle-structure is $(\square \circlearrowright \circlearrowleft \clubsuit \text{¥} \circlearrowleft \triangle \heartsuit \divideontimes)$; $OL_\beta =$ one 3-cycle, and $EL_\beta =$ one 4-cycle. So

$$\text{Sgn}(\beta) = [-1]^{EL_\beta} = -1;$$

so β is an odd permutation.

Counting transpositions. One transposition sequence is:



Five transpositions; so $\text{Sgn}(\beta) = [-1]^5 = -1$.

Counting inversions. If we enumerate $\square=1$, $\clubsuit=2$, $\triangle=3$, $\circlearrowleft=7$, $\text{¥}=6$, $\heartsuit=5$, $\divideontimes=4$. Now our β permutation looks like

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 4 & 1 & 2 \end{array}$$

The inversions are

$$\begin{array}{c} (1, 2), \dots (1, 7), (2, 3), \dots (2, 7), (3, 4), \dots (3, 7), \\ (4, 6), (4, 7), (5, 6), (5, 7). \end{array}$$

So the inversion number is $6 + 5 + 4 + 2 + 2 = 19$.

Let's try a different numbering: $\circlearrowleft=2$, $\text{¥}=4$, $\heartsuit=6$, $\square=1$, $\clubsuit=3$ $\triangle=5$, $\divideontimes=7$,

Our β now presents as

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 6 & 7 & 5 \end{array}$$

With this enumeration, the inversions are

$$(1, 4), (2, 4), (3, 4), (5, 7), (6, 7).$$

so the inversion number is 5.