Linear Algebra GenFacts

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ABSTRACT: Centralizers. Determinant facts (not done).

Vandermonde Matrix

(This works over every integral domain. And we can freely pass to its field-of-quotients.) An $[N+1]$-tuple $\mathbf{b} := (\beta_0, \beta_1, \ldots, \beta_N)$ determines a product

1: $$G(\mathbf{b}) := \prod_{0 \le i < j \le N} [\beta_j - \beta_i]$$

and the $[N+1] \times [N+1]$ ***Vandermonde matrix***

2: $$\mathsf{V}_{\mathbf{b}} := \begin{bmatrix} 1 & \beta_0 & \beta_0^2 & \ldots & \beta_0^N \\ 1 & \beta_1 & \beta_1^2 & \ldots & \beta_1^N \\ 1 & \beta_2 & \beta_2^2 & \ldots & \beta_2^N \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \beta_N & \beta_N^2 & \ldots & \beta_N^N \end{bmatrix}.$$

3: Vandermonde Lemma. *For all tuples* $\mathbf{b}$,

3i: $$\mathrm{Det}(\mathsf{V}_{\mathbf{b}}) = G(\mathbf{b}).$$ ◊

Proof. We induct on N. For $N=0$, remark that

$$\mathrm{Det}([1]) = 1 = \text{Empty product} = G((\beta_0)).$$

Fix an $N \ge 1$. Consider the $N \times N$ submatrix

$$\mathsf{S} := \begin{bmatrix} \beta_1 & \beta_1^2 & \ldots & \beta_1^N \\ \beta_2 & \beta_2^2 & \ldots & \beta_2^N \\ \vdots & \vdots & & \vdots \\ \beta_N & \beta_N^2 & \ldots & \beta_N^N \end{bmatrix}.$$

From its first row, we can factor out a β_1; from its second, a β_2. Continuing, gives

$$\mathrm{Det}(\mathsf{S}) = p \cdot \mathrm{Det}(\mathsf{V}_{(\beta_1,\ldots,\beta_N)}),$$

where $p := \beta_1 \cdot \beta_2 \cdots \beta_N$. By induction, then,

3ii: $$\mathrm{Det}(\mathsf{S}) = p \cdot G\Big((\beta_1, \ldots, \beta_N)\Big) \stackrel{\text{note}}{=\!=} G\Big((0, \beta_1, \ldots, \beta_N)\Big).$$

Equality of polys. There is no true loss-of-generality in assuming that none of $\beta_1, \ldots, \beta_N$ is zero. (Left to the Reader.) Define two polys

$$f(x) := \mathrm{Det}(\mathsf{V}_{(x,\beta_1,\ldots,\beta_N)});$$
$$g(x) := G((x, \beta_1, \ldots, \beta_N)).$$

Our goal is $f(\beta_0) \stackrel{?}{=} g(\beta_0)$. Since f, g each have degree$\le N$ [exercise!], we need but establish equality $f(x)=g(x)$, for x at $N+1$ distinct places. (This implication obtains, since we work over an integral domain). We will use the values $x = 0, \beta_1, \ldots, \beta_N$.

Certainly $g(\beta_1) = 0$, by (1). And $f(\beta_1)$ is zero, since the corr. Vandermonde matrix has duplicated rows. Ditto $\beta_2, \ldots, \beta_N$.

Applying cofactors along the top row of $\mathsf{V}_{(0,\beta_1,\ldots,\beta_N)}$ gives $\mathrm{Det}(\mathsf{V}_{(0,\beta_1,\ldots,\beta_N)}) = \mathrm{Det}(\mathsf{S})$. I.e, $f(0) = \mathrm{Det}(\mathsf{S})$. Thus

$$f(0) = G\Big((0, \beta_1, \ldots, \beta_N)\Big) \stackrel{\text{by defn}}{=\!=} g(0).$$ ♦

Discriminant. Here is an application. Define

$$h(x) := \prod_{j=0}^{N} [x - \beta_j];$$

this is the above $g(x)$ up to multiplying by ± 1. Counted with multiplicity, the roots of h are $\beta_0, \ldots, \beta_N$.

Set $\mathbf{c} := (\beta_0, \ldots, \beta_N)$. Courtesy (1) & (3i) we have

4:
$$\operatorname{Discr}(h) \overset{\text{def}}{=} G(\mathbf{c})^2 = \operatorname{Det}(\mathsf{V}_{\mathbf{c}})^2 \overset{\text{note}}{=} \operatorname{Det}(\mathsf{V}_{\mathbf{c}}^{\mathsf{t}} \cdot \mathsf{V}_{\mathbf{c}}).$$

Here, $\mathsf{V}_{\mathbf{c}}^{\mathsf{t}}$ is the transpose of $\mathsf{V}_{\mathbf{c}}$. For $k = 0, 1, 2, \ldots$, define the sum

$$\mu^{(k)} := \mu^{(k)}_{\mathbf{c}} := \sum_{j \in [1..N]} [\beta_j]^k.$$

For illustrative purposes, suppose $N = 25$ and write $\mathbf{c} = (a, b, \ldots, z)$. Then

4′:
$$\mathsf{V}_{\mathbf{c}}^{\mathsf{t}} \cdot \mathsf{V}_{\mathbf{c}} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ a & b & \ldots & z \\ a^2 & b^2 & \ldots & z^2 \\ \vdots & & & \\ a^N & b^N & \ldots & z^N \end{bmatrix} \cdot \begin{bmatrix} 1 & a & a^2 & \ldots & a^N \\ 1 & b & b^2 & \ldots & b^N \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z & z^2 & \ldots & z^N \end{bmatrix} = \begin{bmatrix} \mu^{(0)} & \mu^{(1)} & \mu^{(2)} & \ldots & \mu^{(N)} \\ \mu^{(1)} & \mu^{(2)} & \mu^{(3)} & \ldots & \mu^{(N+1)} \\ \mu^{(2)} & \mu^{(3)} & \mu^{(4)} & \ldots & \mu^{(N+2)} \\ \vdots & & & & \\ \mu^{(N)} & \mu^{(N+1)} & \mu^{(N+2)} & \ldots & \mu^{(N+N)} \end{bmatrix}.$$

Observe Unfinished: as of 26Aug2023 I don't know where I meant to take this. □

Centralizer

Consider field F, ring $\mathcal{R} := \mathrm{MAT}_{N\times N}(\mathsf{F})$ and group $\mathcal{G} := \mathrm{GL}_N(\mathsf{F})$. Use $\mathbf{I}$ for the identity matrix. Let $\Delta_{\mathsf{i},\mathsf{j}}$ be the all-zero matrix *except* for a 1 in position (i, j).

5: Lemma. *Suppose matrix* $\mathsf{M} \in \mathcal{R}$ *commutes with every member of* $\mathcal{G}$. *Then* M *is a multiple of* $\mathbf{I}$. ◊

Proof. WLOG $N \geq 2$. Write M in rows and cols:

†:
$$\mathsf{M} =: \begin{bmatrix} \rightarrow \mathbf{r}_1 \rightarrow \\ \vdots \\ \rightarrow \mathbf{r}_N \rightarrow \end{bmatrix} =: \begin{bmatrix} \downarrow & \downarrow & & \downarrow \\ \mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Let α be the $(1,1)$-entry of M. ISTShow that each $\mathbf{r}_{\mathsf{j}}$ equals $(0, \overset{\mathsf{j}-1}{\ldots}, 0, \alpha, 0, \ldots, 0)$. I.e that $\mathbf{r}_{\mathsf{j}} = \alpha \mathbf{e}_{\mathsf{j}}$.

Assumption, "$\mathsf{M} \leftrightarrows \Delta$". We show $\mathbf{r}_3 = \alpha \mathbf{e}_3$. We first assume that M commutes with $\Delta := \Delta_{1,3}$, even though Δ is not invertible. The 1st-row of product $\Delta\mathsf{M}$ is $\mathbf{r}_3$; all other rows are zero. The 3rd-column of $\mathsf{M}\Delta$ is $\mathbf{c}_1$; all other columns are zero. Since $\Delta\mathsf{M} \overset{\text{must}}{=} \mathsf{M}\Delta$, this 1st-row and 3rd-column must be all-zero except for a value, call it β, at their common intersection. So $\mathbf{r}_3 = \beta \mathbf{e}_3$ and (viewed as "vertical" tuples) $\mathbf{c}_1$ equals $\beta \mathbf{e}_1$. The equality $\mathbf{c}_1 = \beta \mathbf{e}_1$ means that β equals α.

Back to Reality. Fix a $\mathsf{j} \neq 1$. So $\mathbf{I} + \Delta$ is invertible, where $\Delta := \Delta_{1,\mathsf{j}}$.

Thus $[\mathbf{I} + \Delta]\mathsf{M} = \mathsf{M}[\mathbf{I} + \Delta]$ and so $\Delta\mathsf{M} = \mathsf{M}\Delta$. As above, then, $\mathbf{r}_{\mathsf{j}} = \alpha \mathbf{e}_{\mathsf{j}}$. So we just need to show that the only non-zero entry in the first row, $\mathbf{r}_1$, is its first entry. But the above argument applied to $\Delta := \Delta_{2,1}$ implies that. ♦

Let Rent(U) for the Ring centralizer of M; the set of $\mathsf{A} \in \mathcal{R}$ which commute with U. Evidently Rent(U) is a subring (indeed, a sub-F-algebra) of $\mathcal{R}$. And it clearly includes the F-algebra of polymonials in U.

Ques. *Q1.* For what U is Rent(U) just the polynomials? For the identity matrix, Rent($\mathbf{I}$) is $\mathcal{R}$.

Now suppose F has a topology under which addition and multiplication are cts. This puts a topology on $\mathrm{MAT}_{N\times N}(\mathsf{F})$ (component-wise) under which matrix-mult is cts. Thus Rent(U) is a closed set. Under what conditions is the closure of Poly(U)? And when is this closure properly bigger than Poly(U)? □

Remark. Over field $\mathbb{Z}_2$, notice that matrix $\mathsf{U} := \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is not only non-invertible, but worse: It cannot be made invertible even by adding a multiple of $\mathbf{I}$. □

Indexed

Consider a (finite, usually) set S, and another set R (which is usually cring), call a map $\mathsf{A}: [S \times S] \to R$ a "(square) matrix ***indexed by*** S".

Kronecker product

Consider two square matrices:

$$\mathsf{A} = \big(\alpha_{\ell,\ell'}\big) \text{ indexed by } [0..L)\,;$$
$$\mathsf{B} = \big(\beta_{m,m'}\big) \text{ indexed by } [0..M)\,.$$

The ***Kronecker product*** $\mathsf{K} := \text{Kron}(\mathsf{A},\mathsf{B})$ is indexed by $[0..L) \times [0..M)$. Its entry at "row" (ℓ, m) and "column" (ℓ', m') is the product $\alpha_{\ell,\ell'} \cdot \beta_{m,m'}$.

6: Theorem. *With notation from above,*

$$\text{Det}(\mathsf{K}) \;=\; \text{Det}(\mathsf{A})^M \cdot \text{Det}(\mathsf{B})^L \qquad \diamond$$

Proof. Apply elementary row operations (ER-Ops) to A, and see that this corresponds to ER-Ops on K. Now do the same for B and K. Let $\widehat{\mathsf{A}}, \widehat{\mathsf{B}}, \widehat{\mathsf{K}}$ be the RREF (reduced row-echelon form) of A, B, K, respectively. I think we will get that

$$\widehat{\mathsf{K}} \;=\; \text{Kron}(\widehat{\mathsf{A}}, \widehat{\mathsf{B}})\,. \qquad ♦$$

Uniqueness of RREF

09Oct2010: This proof is cryptic; improve it.

Below, B and $\widehat{\mathsf{B}}$ are $K{\times}N$ matrices in *reduced row-echelon form*. Use $\mathbf{r}_i$ and $\widehat{\mathbf{r}}_i$ to denote their i^{th} rows.

7: RREF-is-Unique Thm. *If a $K{\times}N$ matrix is row-equivalent to both $\widehat{\mathsf{B}}$ and B, then $\widehat{\mathsf{B}} = \mathsf{B}$.* $\diamond$

Proof. Row-equivalence preserves row-span, and so the thm will follow from Lemma 8. ♦

8: Lemma. *Suppose that $\widehat{\mathsf{B}} \neq \mathsf{B}$. Let δ be the smallest row-index for which $\widehat{\mathbf{r}}_\delta \neq \mathbf{r}_\delta$. Then either*

$$*: \qquad \widehat{\mathbf{r}}_\delta \;\notin\; \text{RowSpn}(\mathsf{B}) \quad \textit{or}$$
$$\widehat{*}: \qquad \mathbf{r}_\delta \;\notin\; \text{RowSpn}(\widehat{\mathsf{B}})\,. \qquad \diamond$$

Proof. WLOG the first-rows disagree, i.e $\delta = 1$. WLOG $\widehat{\mathsf{B}}(1,1)$ and $\mathsf{B}(1,1)$ are pivot-ptns. (Exercise.)

The difference row-vector

$$\mathbf{d} \;:=\; \widehat{\mathbf{r}}_1 - \mathbf{r}_1$$

is not all-zero. There is no true loss of generality in assuming that column-5 is the first non-zero entry in $\mathbf{d}$. So

$$\mathbf{d} \;=\; [\,0\;0\;0\;0\;\widehat{x}{-}x\;?\;?\;\ldots\;?\,]$$

where $x := \mathbf{r}_1(5)$. And $\widehat{x} \neq x$.

Suppose that $(*)$ is false. Then there are row-indices $2 \geq \mathsf{i}_1 > \mathsf{i}_2 > \mathsf{i}_3 > \ldots$ and *non-zero* scalars α_{i_k} so that

$$**: \qquad \mathbf{d} \;=\; \alpha_{\mathsf{i}_1}\mathbf{r}_{\mathsf{i}_1} + \alpha_{\mathsf{i}_2}\mathbf{r}_{\mathsf{i}_2} + \ldots\,.$$

Moreover, this sum is <u>*not empty*</u>, since $\mathbf{d} \neq \mathbf{0}$.

All the rows of $(**)$ must have their pivot-cols at ≥ 5, since cols $1, \ldots, 4$ of $\mathbf{d}$ are 0.

Thus the pivot-col of $\mathbf{r}_{\mathsf{i}_1}$ *must* be 5, since $\mathbf{d}(5)$ is not zero. But this means that $\boxed{x = 0}$.

Doing the same for the hatted rows. FTSOC, suppose that also $(\widehat{*})$ fails. Then the same reasoning shows that there are row-indices $2 \geq j_1 > j_2 > j_3 > \ldots$ and non-zero scalars β_{j_k} so that

$$\widehat{**}: \qquad \mathbf{d} \;=\; \beta_{j_1}\widehat{\mathbf{r}}_{j_1} + \beta_{j_2}\widehat{\mathbf{r}}_{j_2} + \ldots\,.$$

As before, the pivot-col of $\widehat{\mathbf{r}}_{j_1}$ <u>must</u> be 5. Hence $\boxed{\widehat{x} = 0}$. And this ※s that $\widehat{x} \neq x$. ♦

Ques. *Q1.* If $\widehat{\mathsf{B}}$ and B have the same row-nullspace, must they be equal? □