Staple!

Name: ______________________ **Ord:** ___ ___

Sets and Logic MHF3202 7860 | **IndividualOP-D** | Prof. JLF King Wedn 17Apr2024

This IOP is due **2PM, Thurs., 25Apr2024**, slid *completely* under my office door, 402 LITTLE HALL.

Print this sheet, which is "Page 1/N", and number your write-up as "page 2/N", "page 3/N" ..."Page N/N".

Your 4 essay(s) must be TYPED, *and* (if possible) *Double spaced. Use the* Print/Revise *cycle to produce good, well thought out, essays. Start each essay on a* **new** *sheet of paper. Do* **not** *restate the problem; just solve it.*

D1: Interval-of-integers $\mathbf{J} := [201..300)$ has 99 elements. A subset $S \subset \mathbf{J}$ is ***Big*** if $|S| = 51$. Subset $S \subset \mathbf{J}$ is ***Perfect*** if there exist *distinct* members $x,y \in S$ st. $x + y = 500$.

Prove that Big $\Rightarrow$ Perfect. [*Hint:* PHP. *Carefully* specify what your pigeon-holes are.]

D2: [For free: Union Thm: *A countable union of countable-sets is countable.* Also, Finite-subset Thm: *The collection of **finite** subsets of a countable set, is countable.* If needed, use $\mathcal{P}_{\text{Fin}}(S)$ for the collection of *finite* subsets of a set S, and use $\mathcal{P}_\infty(S)$ for the collection of **infinite** subsets of S.] Below, a ***blip*** is an *infinite* set of natnums. A ***family***, $\mathcal{F}$, is a set [*not* a multiset] of blips, i.e, $\mathcal{F} \subset \mathcal{P}_\infty(\mathbb{N})$.

i Suppose, $\forall B,C \in \mathcal{F}$, that $[B \neq C] \Rightarrow [B \cap C = \varnothing]$. Construct, with proof, an *injection* $g{:}\mathcal{F} \hookrightarrow \mathbb{N}$, showing that each such family, $\mathcal{F}$, must only be countable. ii Weaken the hypothesis on $\mathcal{F}$ to:

$$\forall B,C \in \mathcal{F}: \quad [B \neq C] \Rightarrow |B \cap C| \leq 1\,.$$

Prove that each such $\mathcal{F}$ is tiny; only countable.

Weaken to $[B \neq C] \Rightarrow |B \cap C| \leq 2$, yet still prove $\mathcal{F}$ countable. Weaken further to $[B \neq C] \Rightarrow |B \cap C| \leq 3$ and prove $|\mathcal{F}| \leq \aleph_0$ still holds. *Generalize!*

iii [Creative; A converse.] Construct a *specific* uncountable family $\mathcal{U} \subset \mathcal{P}_\infty(\mathbb{N})$, so that:

For all distinct $B,C \in \mathcal{U}$: Intersection $B \cap C$ is finite.

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: ....

D3: Consider $\mathbf{S}$, a set of 120 Students. Coincidentally, exactly 10 were born in January, exactly 10 in February, exactly 10 in March,..., exactly 10 in December.

Astonishingly, *exactly* 10 were born in *each* of the twelve years 2001, 2002, ..., 2012.

i Prove that *there exists* a set, T, of twelve Students whose birth-months are *all* twelve months, and whose birth-years are *all* twelve years.

ii Some N many Students depart, leaving a smaller group $\mathbf{S}'$ with only $|\mathbf{S}'| = 120 - N$ many Students. With proof, what is the largest value of N where there still *must* exist a 12-set $T' \subset \mathbf{S}'$ representing all twelve months and all twelve years? This $N =$ ....

D4: A polygamous community comprises 90 women and 91 men. Each man has at least one wife. Prove

†: *There is a married couple such that the wife has more husbands than the husband has wives.*

[The people are women and men, but the problem will use "girls" and "boys", so that I can use letters W and H for wives and husbands.]

Guided soln: Use $\mathbb{G}$ for set of 90 girls and $\mathbb{B}$ for the set of 91 boys. For a $\mathbf{g} \in \mathbb{G}$, let $H_\mathbf{g}$ be the number of husbands she has. So if *Ann* is married to *Tom*, *Sid* and *Abe*, then $H_{Ann} = 3$. For $\mathbf{b} \in \mathbb{B}$, use $W_\mathbf{b}$ for the number of wives that boy $\mathbf{b}$ has.

Use a double-colon to indicate marriage, i.e, *Ann*::*Tom* indicates that *Ann* and *Tom* are married to each other.

Possible error: You might think the "*average girl*" has $\frac{91}{90}$ husbands, whereas the "*average boy*" has $\frac{90}{91}$ wives. Even if true, this does *not prove* that there is a married-to-each-other couple $\mathbf{g}{::}\mathbf{b}$ with $H_\mathbf{g} > W_\mathbf{b}$.

Suggestion: After any preliminary definitions, start your proof with: FTSOContradiction, suppose

‡: *each married couple* $\mathbf{g}{::}\mathbf{b}$ *has* $H_\mathbf{g} \leq W_\mathbf{b}$.

[A ※ can be derived from Hall's Matching Lemma, but a more direct argument exists. Consider small examples, e.g, $|\mathbb{G}| = 3$ and $|\mathbb{B}| = 4$. Is assumption that each-boy-has-at-least-one-wife necessary?]

DO SOMETHING EXTRA: E.g, with N women and $N+2$ men, what can you conclude about the ∃ence of a married couple? What if there are $2N - 1$ men?

Folks, I've had a wonderful time Problem-Solving with you. Stop by in future semesters for Math/chess/coffee.

Cheers, Coun-SELO-r King