Complex Analysis homework.

by *Energetic Plex Student*, April 2023

Books. Use ***FC*** for *First Course in Complex Analysis* by Matthias Beck, Gerald Marchesi, Dennis Pixton & Lucas Sabalka.

Use ***B&C*** for *Brown & Churchill*. The exercise #s in B&C have neither chapter-# nor section-#. So we will disambiguate by using the first-page number where that exercise-block began. Eg. B&C**#8***P.22* is problem #8 in the section starting on P.22 in the 8th edition; the problem is actually on P.23. [And `complex-hw:BC.P22.08-.latex` is the corresponding file.]

Use ***Ahl*** for *Lars Ahlfors* classic *Complex Analysis* text.

Finally, there are a few problems I concocted or altered, labeled ***jk***. □

Standing notation for Integration

The following notation will be used in the sequel. Further below is a statement of the CIF (Cauchy Integral Formula) and the generalized version.

Miscellaneous. Use both Möbius-trn and Möb-trn to abbreciate '*Möbius transformation*'. An LFT (*linear fractional trn*) and cross-ratio are two ways of describing a Möbius-trn.

Recall: For a radius $r>0$, and a point $\mathbf{p}$ in metric space $(\mathbf{X}, \mathsf{m})$ we defined ***open ball***, ***closed ball***, ***sphere*** and ***punctured (open) ball*** as:

$$\begin{aligned}
\mathrm{Bal}_r(\mathbf{p}) &:= \{w \in \mathbf{X} \mid \mathsf{m}(w,\mathbf{p}) < r\};\\
\mathrm{CldBal}_r(\mathbf{p}) &:= \{w \in \mathbf{X} \mid \mathsf{m}(w,\mathbf{p}) \le r\};\\
\mathrm{Sph}_r(\mathbf{p}) &:= \{w \in \mathbf{X} \mid \mathsf{m}(w,\mathbf{p}) = r\};\\
\mathrm{PBal}_r(\mathbf{p}) &:= \{w \in \mathbf{X} \mid 0 < \mathsf{m}(w,\mathbf{p}) < r\}.
\end{aligned}$$ □

Contours. A ***contour*** S is an oriented curve in $\mathbb{C}$ or $\widehat{\mathbb{C}}$. I'll use $\mathsf{L}, \mathsf{C}, \mathsf{S}, \mathsf{D}, \mathsf{A}$ for contours, *usually* using L for a line-segment, and C for a closed contour (a loop, which might self-intersect). A SCC [Simple-Closed-Contour] is closed-contour that does not self-intersect, and is positively oriented.

While C *might* be used for any contour (closed or not), I'll reserve $\mathsf{C}_{\text{subscript}}$ to mean the positively-oriented *circle* about the *origin* with the specified (positive) radius. I.e, $\boxed{\mathsf{C}_r := \mathrm{Sph}_r(0)}$.

For a SCC S, let $\mathring{\mathsf{S}}$ be the (open) region enclosed by S. Let $\widehat{\mathsf{S}} := \mathsf{S} \sqcup \mathring{\mathsf{S}}$, which is closed (indeed, *compact*). E.g, $\mathring{\mathsf{C}}_2 = \mathrm{Bal}_2(0)$ and $\widehat{\mathsf{C}_2} = \mathrm{CldBal}_2(0)$ □

Integrals. An integral on S is $\int_{\mathsf{S}}$. [On a *closed* contour, C, I might use $\oint_{\mathsf{C}}$ to emphasize that C is closed. (Optional, as it sometimes makes the notation too "noisy".)]

Fix a parametrization $\sigma\colon[3,5]\to\mathbb{C}$ of S. A ***contour integral*** of fnc h has form

$$\int_{\mathsf{S}} h = \int_{\mathsf{S}} h(z)\,\mathrm{d}z \stackrel{\text{note}}{=\!=} \int_3^5 h(\sigma(t))\,\sigma'(t)\,\mathrm{d}t\,.$$

In contrast, an ***arclength integral*** [abbrev. ***arclen-int***] is

$$\int_{\mathsf{S}} h(z)\,|\mathrm{d}z| = \int_3^5 h(\sigma(t))\,|\sigma'(t)\,\mathrm{d}t|\,.$$

Since $5 \ge 3$, we can drop the abs-value around the $\mathrm{d}t$, and write the integral as $\int_3^5 h(\sigma(t))\,|\sigma'(t)|\,\mathrm{d}t$.

These integrals satisfy this inequality:

$$*\colon\quad \Big|\int_{\mathsf{S}} h(z)\,\mathrm{d}z\Big| \le \int_{\mathsf{S}} |h(z)|\cdot|\mathrm{d}z| \le \operatorname*{Max}_{z\in\mathsf{S}}|h(z)|\cdot\mathrm{Len}(\mathsf{S})$$

where the length of S is $\mathrm{Len}(\mathsf{S}) = \int_{\mathsf{S}} |\mathrm{d}z|$. Note $(*)$ is analogous to $|a_1 + a_2| \le |a_1| + |a_2| \le \operatorname*{Max}_{j=1,2}|a_j|\cdot 2$.

Consider a point $\mathbf{p}$ *enclosed by a SCC* C. If f is holomorphic on $\widehat{\mathsf{C}}$ [i.e, C *and* the region it encloses] then

$$\text{CIF:}\qquad f(\mathbf{p}) = \frac{1}{2\pi\mathbf{i}}\oint_{\mathsf{C}} \frac{f(z)}{z-\mathbf{p}}\,\mathrm{d}z\,.$$

[CIF = Cauchy Integral Formula.] Differentiating n-times under the integral sign yields

$$\text{CIF}_n\text{:}\qquad f^{(n)}(\mathbf{p}) = \frac{n!}{2\pi\mathbf{i}}\oint_{\mathsf{C}} \frac{f(z)}{[z-\mathbf{p}]^{n+1}}\,\mathrm{d}z\,.$$ □

[When I don't choose to specify the n, I'll use GCIF for Generalized CIF.]

CIF *locally.* For a *point* $\mathbf{p}$, let $\oint_{\mathbf{p}} h(z)\,\mathrm{d}z$ be the integral around $\mathbf{p}$ on a circle whose radius, ε, is small enough that the only h-singularity is (possibly) $\mathbf{p}$. To make specific, h is differentiable on $\mathrm{PBal}_{2\varepsilon}(\mathbf{p})$. □

Matrix convenience. Given two matrices M and K, write $\mathsf{M} \overset{\times\,\text{nzc}}{\frown} \mathsf{K}$ if I can multiply M by some complex $\mu \neq 0$ to produce K. [So '× nzc' stands for 'times non-zero constant'.]

E.g $\begin{bmatrix} 2+\mathbf{i} \\ 3 \end{bmatrix} \overset{\times\,\text{nzc}}{\frown} \begin{bmatrix} 2\mathbf{i}-1 \\ 3\mathbf{i} \end{bmatrix}$ since $\mathbf{i} \cdot \begin{bmatrix} 2+\mathbf{i} \\ 3 \end{bmatrix}$ equals $\begin{bmatrix} 2\mathbf{i}-1 \\ 3\mathbf{i} \end{bmatrix}$. Also, $\begin{bmatrix} -3 \\ 0 \end{bmatrix} \overset{\times\,\text{nzc}}{\frown} \begin{bmatrix} \pi+7\mathbf{i} \\ 0 \end{bmatrix}$. An example with 2×2 matrices is

$$\begin{bmatrix} 3\mathbf{i} & 2 \\ 0 & \mathbf{i}-4 \end{bmatrix} \overset{\times\,\text{nzc}}{\frown} \begin{bmatrix} 3 & -2\mathbf{i} \\ 0 & 1+4\mathbf{i} \end{bmatrix},$$

since $-\mathbf{i}$ times the LH-matrix gives the RH-matrix. □

B&C#3Ex. *P. 12*. Polynomial $P(z) := \sum_{j=0}^{N} \mathbf{c}_j z^j$ has degree $N \geq 1$. Prove there exists $R > 0$ such that

†: $$\left|\frac{1}{P(z)}\right| < \frac{2}{|\mathbf{c}_N| \cdot R^N}, \quad \forall z \text{ with } |z| > R.$$

B&C#8 *P. 22*. Non-zero complex P,Q have equal moduli IFF $\exists \mu,\nu \in \mathbb{C}$ with $P = \mu\nu$ and $Q = \mu\overline{\nu}$. [When these two equalities hold, say (μ,ν) ***makes*** (P,Q).]

Chapter 2

FC#2.16$^{P.}$*30*. Prove, if $f(z)$ is given by a polynomial in z, that f is entire.

What can you say if $f(z)$ is given by a polynomial in $x = \text{Re}(z)$ and $y = \text{Im}(z)$?

FC#2.17$^{P.}$*30*. (Dis)Prove:

A: If u and v are real valued and continuous, then $f = \mathsf{u} + \mathbf{i}\mathsf{v}$ is continuous

B: If u and v are $\mathbb{R}\times\mathbb{R}$-differentiable then $f := \mathsf{u}+\mathbf{i}\mathsf{v}$ is (complex) differentiable.

FC#2.22$^{P.}$*30*. Suppose f is entire and can be written as $f(z) = \mathsf{u}(x) + \mathbf{i}\mathsf{v}(y)$, i.e, the real part of f depends only on $x = \mathrm{Re}(z)$ and the imaginary part of f depends only on $y = \mathrm{Im}(z)$. Prove that $f(z) = az + \beta$ for some $a \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

FC#2.23$^{P.}$*30*. Suppose f is entire, with real and imaginary parts u and v satisfying $\mathsf{u}() \cdot \mathsf{v}() \equiv 3$. Prove f is constant.

FC#2.25b *P.30*. For $\mathsf{u}(x, y) \coloneqq \cosh(y) \cdot \sin(x)$ find a real-valued $\mathsf{v}(x, y)$ st $\mathsf{u} + \mathsf{v}\mathbf{i}$ in holomorphic in some region.

FC#2.25c *P.30*. For $\mathsf{u}(x, y) \coloneqq 2x^2 + x + 1 - 2y^2$ find a real-valued $\mathsf{v}(x, y)$ st $\mathsf{u} + \mathsf{v}\mathbf{i}$ in holomorphic in some region.

FC#2.25d$^{P.}$*30*. For $\mathsf{u}(x,y) := \frac{x}{x^2+y^2}$ find a real-valued $\mathsf{v}(x,y)$ such that $\mathsf{u} + \mathsf{v}\mathbf{i}$ in holomorphic in some region. Maximize that region.

FC#2.27$^{P.}$*30*. With A,B,$C \in \mathbb{R}$, consider the homogeneous quadratic $\mathsf{u}(x,y) := Ax^2 + Bxy + Cy^2$. Find an IFF condition on A,B,C making u harmonic.

With u harmonic, compute the complex number M such that u is the real part of $z \mapsto Mz^2$.

Chapter 3

FC#3.03$^{P.}$*48*. For *reals* α, P, Q, ω, consider equation

$$\dagger: \quad \alpha[x^2 + y^2] + Px + Qy + \omega \;=\; 0$$

in $\mathbb{R}\times\mathbb{R}$. Show that ($\dagger$) describes a ***gencircle*** [i.e, a *circle*-or-*line*; a ***generalized-circle***] IFF

$$*: \quad P^2 + Q^2 \;>\; 4\alpha\omega\,.$$

FC#3.05$^{P.}$*48*. Prove that each Möbius transformation $f(z) = \frac{az+b}{cz+d}$ different from the identity-map, has at most two fixed-points. [Recall a *Möbius-trn* requires that *determinant* $\Delta \neq 0$, where $\Delta := ad - bc$.]

FC#3.09*P.48*. Fix $\mathbf{v} \in \mathbb{C}$ with $|\mathbf{v}| < 1$ and consider

†: $$f_{\mathbf{v}}(z) := \frac{z - \mathbf{v}}{1 - \overline{\mathbf{v}}z}.$$

a: Prove $f_{\mathbf{v}}$ is a Möbius transformation.

b: Show $f_{\mathbf{v}}^{-1} = f_{-\mathbf{v}}$.

c: Prove $f_{\mathbf{v}}$ maps the unit ball $\mathbb{B}$ to itself bijectively.

FC#3.16*P.48*. Find a Möbius transformation $h()$ that preserves C_1, with $h(0) = \frac{1}{2}$.

FC#3.23$^{P.}$*48*. Given $A \in \mathbb{R} \setminus \{0\}$, let L be the $y = A$ line. Show that the image of L under inversion is the circle with center $\frac{-\mathbf{i}}{2A}$ and radius $\frac{1}{2A}$.

Remark. A small oversight: The radius is $\frac{1}{2 \cdot |A|}$.

Parameterize the image-circle, C, by $F(t) := \frac{1}{t + A\mathbf{i}}$. [Let's solve the problem *without* knowledge of the center & radius that the problem gave.]

One approach: Use F to compute the curvature of C; its reciprocal is the radius.

Another approach: Compute the C tangent-line and orthogonal line at different points $F(t_0)$ and $F(t_1)$. The intersection of the two ortho-lines is C's center. □

FC[#]**3.27**[P.]*48*. In $\mathbb{R}^3$, consider the plane H determined by $x + y - z = 0$. What is a unit normal-vector to H? In Riemann-sphere $\mathbb{RS}$, compute the image of $\mathsf{H} \cap \mathbb{RS}$ under stereographic projection $\mathcal{S}()$.

Chapter 4

FC#4.1d$^{P.}$*68*. Compute the arclength, L, of cycloid $\sigma(t) := t - \mathbf{i}e^{-\mathbf{i}t}$ for one roll, $0 \le t \le 2\boldsymbol{\pi}$.

Predictions. [First, we sketch the cycloid arch. At multiples of $2\boldsymbol{\pi}$, the parametrization must have zero-vel. *not* because we differentiated σ, but rather because that is where the moving wheel-point *touches the ground*, and the ground ain't moving'.]

The arch has symmetry about the vertical line at $t{=}\boldsymbol{\pi}$. A picture shows that the half-arch, $\frac{L}{2}$, should satisfy $\boxed{\boldsymbol{\pi} < \frac{L}{2} < 1 + \boldsymbol{\pi}}$ Using the diagonal, we obtain a better lower bound; $\boxed{\sqrt{1^2 + \boldsymbol{\pi}^2} < \frac{L}{2}}$. □

FC#4.05$^{P.}$*68*. Around circle $\mathsf{C} := \mathsf{C}_R$, integrate: $\varphi(z) := z^2 - 2z + 3$, $f(z) := z + \overline{z}$, $g(z) := 1/z^4$ and $h(x + \mathbf{i}y) := xy$.

Prelim. Parametrize C by $\boxed{\sigma(t) := R\mathrm{e}^{\mathbf{i}t}}$. □

FC#4.11$^{P.}$*68*. Let $\mathcal{I}(k) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathbf{i}kt}\,\mathrm{d}t$.

i: Show $\mathcal{I}(0) = 1$.

ii: Show $\mathcal{I}(k) = 0$ for each non-zero integer k.

iii: Compute $\mathcal{I}(\frac{1}{2})$.

FC#4.23[P.68]**.** Prove that $\sim_G$ is an equivalence relation.

FC#4.25[P.68]**.** Prove that each closed path is $\mathbb{C}$-contractible. Prove that each two closed paths are $\mathbb{C}$-homotopic.

Several FC probs use the same idea. Recall $\boxed{\mathsf{C}_r := \mathrm{Sph}_r(0)}$.

1: Recip-polynomial lemma. *For a polynomial $f(z)$ of degree $N \geq 2$ take an R large enough that all of f's roots lie inside C_R. Then*

$$J := \oint_{\mathsf{C}_R} \frac{1}{f(z)}\,\mathrm{d}z = 0\,. \qquad \lozenge$$

Proof. There is some $\kappa > 0$ so that, for all large r, our $|f(z)| > |z|^N \cdot \kappa$ when $z \in \mathsf{C}_r$. For $r \geq R$, the CHT (Cauchy Homotopy Thm) gives $\mathsf{C}_R \sim \mathsf{C}_r$ on the punctured plane. Thus

$$|J| \stackrel{\mathrm{CHT}}{=\!=} \left|\oint_{\mathsf{C}_r} \frac{1}{f(z)}\,\mathrm{d}z\right| \leq \mathrm{Len}(\mathsf{C}_r) \operatorname*{Max}_{z \in \mathsf{C}_r} \frac{1}{|f(z)|}$$
$$\leq 2\pi r \cdot \frac{1}{r^N \cdot \kappa} = \frac{\mathrm{Const}}{r^{N-1}}\,.$$

This last $\to 0$ as $r \nearrow \infty$, since $N \geq 2$. ♦

FC#4.29[P.68]**.** Show that $\displaystyle\int_{\mathsf{C}_2} \frac{\mathrm{d}z}{z^3 + 1}$ is zero.

FC#4.35[P.68]**.** For $r = 1,3,5$, compute

$$J_r := \oint_{\mathsf{C}_r} \frac{\mathrm{d}z}{z^2 - 2z - 8}\,\mathrm{d}z\,.$$

FC#4.34[P.68]**.** Let $\mathsf{S}_r := \mathrm{Sph}_r(\text{-}2\mathbf{i})$. For each $r \neq 1,3$, compute $J_r := \displaystyle\oint_{\mathsf{S}_r} \frac{\mathrm{d}z}{z^2 + 1}$.

FC#4.37$^{P.}$*68*. With $\mathsf{C}_r := \mathrm{Sph}_r(0)$ and $\mathsf{S} := \mathrm{Sph}_2(-1)$, compute these four integrals: [CIF solves all four.]

a: $\displaystyle\oint_{\mathsf{S}} \frac{z^2}{4 - z^2}\, \mathrm{d}z$,

b: $\displaystyle\oint_{\mathsf{C}_1} \frac{\sin(z)}{z}\, \mathrm{d}z$.

c: $\displaystyle\oint_{\mathsf{C}_2} \frac{\exp(z)}{z\,[z - 3]}\, \mathrm{d}z$

d: $\displaystyle\oint_{\mathsf{C}_4} \frac{\exp(z)}{z\,[z - 3]}\, \mathrm{d}z$

FC#**4.30**$^{P.}$*68*. Compute $J := \int_0^{2\pi} \frac{1}{2 + \sin(\theta)} \,\mathrm{d}\theta$.

jk#**Trig-CoV**$^{P.}$. Consider integral

$$*: \qquad J := \int_0^{2\pi} \frac{\cos(\theta)\cdot\cos(3\theta)}{2 + \sin(2\theta)} \,\mathrm{d}\theta$$

Integrating around the unit circle, J equals $\oint \frac{f(z)}{g(z)}\,\mathrm{d}z$, for which polynomials f and g ? [*Hint:* CoV $z = \mathrm{e}^{\mathbf{i}\theta}$.]

Recall. CoV $z := \mathrm{e}^{\mathbf{i}\theta}$ transforms $[0, 2\pi]$ into U, the unit-circle. Moreover, for $k \in \mathbb{Z}$:

$$\mathrm{d}\theta = \frac{\mathrm{d}z}{\mathbf{i}z},$$

2a: $$\cos(\theta) = \tfrac{1}{2}[z + \tfrac{1}{z}] = \frac{z^2+1}{2z}, \quad \cos(k\theta) = \frac{z^{2k}+1}{2z^k},$$

$$\sin(\theta) = \tfrac{1}{2\mathbf{i}}[z - \tfrac{1}{z}] = \frac{z^2-1}{2\mathbf{i}z}, \quad \sin(k\theta) = \frac{z^{2k}-1}{2\mathbf{i}z^k}.$$

Thus a $\int_0^{2\pi}$ integral of a rational function of $\cos(k\theta)$ and $\sin(\ell\theta)$ is transformed, by the CoV, into a $\int_{\mathsf{U}}$ integral of a rational fnc of z. Factoring the denominator gives the poles of the integrand, so we can apply CIF, equivalently, the Residue thm. □

FC#4.31[P] *68*. Prove for, $0 \le \mathbf{r} < 1$, that

$$\dagger: \qquad \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - \mathbf{r}^2}{1 - 2\,\mathbf{r}\cos(\theta) + \mathbf{r}^2}\,d\theta \quad = \quad 1\,.$$

[The function $\mathsf{P}_{\mathbf{r}}(\theta) := \frac{1-\mathbf{r}^2}{1-2\mathbf{r}\cos(\theta)+\mathbf{r}^2}$ is the ***Poisson kernel***. It plays an important role in the world of harmonic functions, as in Exercise **#6.13**[P] *89*.]

Proof. With $\mathsf{U} := \mathsf{C}_1$ the unit circle, we seek to write the given (†)-integral as a $\oint_{\mathsf{U}}$ integral, then apply CIF.

Letting $z := e^{i\theta}$, recall $2\cos(\theta) = z + z^{-1}$, making the (†)-integrand

$$\frac{1 - \mathbf{r}^2}{1 - \mathbf{r}\,[z + z^{-1}] + \mathbf{r}^2} \overset{\times\frac{z}{z}}{=\!=} \frac{[1 - \mathbf{r}^2]\,z}{z - \mathbf{r}\,[z^2+1] + \mathbf{r}^2 z}$$

$$\overset{\text{factor denom}}{=\!=\!=} \frac{[1 - \mathbf{r}^2]\,z}{[z - \mathbf{r}]\cdot[1 - \mathbf{r}z]}\,.$$

Equality $\frac{dz}{d\theta} = \mathbf{i}\cdot z$ gives $d\theta = \frac{1}{\mathbf{i}z}\,dz$, rewriting the (†)-integral as

$$\oint_{\mathsf{U}} \frac{[1 - \mathbf{r}^2]\cdot z}{[z - \mathbf{r}][1 - \mathbf{r}z]}\cdot\overbrace{\frac{1}{\mathbf{i}z}\,dz}^{d\theta} = \frac{1}{\mathbf{i}}\oint_{\mathsf{U}} \frac{1 - \mathbf{r}^2}{[z - \mathbf{r}][1 - \mathbf{r}z]}\,dz$$

$$= \frac{1}{\mathbf{i}}\oint_{\mathsf{U}} \frac{V(z)}{z - \mathbf{r}}\,dz\,,$$

where $V(z) := \frac{1 - \mathbf{r}^2}{1 - \mathbf{r}z}$. The Cauchy Integral Formula now gives

$$\text{LhS}(\dagger) = \frac{1}{2\pi}\cdot\frac{1}{\mathbf{i}}\oint_{\mathsf{U}} \frac{V(z)}{z - \mathbf{r}}\,dz \overset{\text{CIF}}{=\!=} V(\mathbf{r}) \overset{\text{note}}{=\!=} 1\,.$$ ♦

3: Poisson kernel. With $\mathbb{P} := \mathbb{R}\times\mathbb{R}$ be the plane, $\mathbb{P}^\circ := \mathbb{P} \setminus \{(1,0)\}$ the punctured plane, interpret Poisson kernel $\mathsf{P}_{\mathbf{r}}(\theta)$ in *polar coordinates*. We argue, further below, that

$$\text{3a:} \qquad \mathsf{P}_{\mathbf{r}}(\theta) \overset{z = \mathbf{r}e^{i\theta}}{=\!=\!=} \operatorname{Re}\Big(\frac{1+z}{1-z}\Big)\,.$$

Hence we view the Poisson kernel as a map $\mathbb{P}^\circ \to \mathbb{R}$. □

jk#Poisson kernel is harmonic[P]. Prove that Poisson kernel $(\mathbf{r},\theta) \mapsto \mathsf{P}_{\mathbf{r}}(\theta)$, interpreted as a polar-coordinate map $\mathbb{P}^\circ \to \mathbb{R}$, is harmonic.

Proof. We *could* apply the polar Laplace operator

$$\text{3b:} \qquad \Delta \mathsf{u} = \tfrac{1}{\mathbf{r}}\mathsf{u}_{\mathbf{r}} + \mathsf{u}_{\mathbf{rr}} + \tfrac{1}{\mathbf{r}^2}\mathsf{u}_{\theta\theta}\,,$$

but there is a shorter, elegant approach.

Fnc $z \mapsto \frac{1+z}{1-z}$ is analytic on $\mathbb{P}^\circ$, so $H(z) := \operatorname{Re}(\frac{1+z}{1-z})$ is harmonic, making $H(\mathbf{r}e^{i\theta}) \overset{?}{=} \mathsf{P}_{\mathbf{r}}(\theta)$ our goal. Since

$$\frac{1+z}{1-z} = \frac{1+z}{1-z}\cdot\frac{1-\overline{z}}{1-\overline{z}} = \frac{1 - z\overline{z} + [z - \overline{z}]}{1 + z\overline{z} - [z + \overline{z}]}\,,$$

substitution $z = \mathbf{r}e^{i\theta}$ with $\mathbf{r},\theta$ real, produces

$$\frac{1+z}{1-z} = \frac{1 - \mathbf{r}^2 + 2\cdot\mathbf{i}\sin(\theta)}{1 + \mathbf{r}^2 - 2\cdot\cos(\theta)}\,.$$

Taking real-parts finishes the proof. ♦

Harmonic with bndry condition. On unit circle U, fix a cts fnc $f:\mathsf{U}\to\mathbb{C}$. Define ***companion fnc*** $\mathrm{I}_f:[\mathbb{P}\setminus\mathsf{U}]\to\mathbb{C}$ by

3c:
$$\mathrm{I}_f(\mathbf{r}\mathrm{e}^{\mathbf{i}\theta}) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\mathsf{P}_{\mathbf{r}}(\theta-t)\cdot f(\mathrm{e}^{\mathbf{i}t})\,\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1\;-\;\mathbf{r}^2}{1\;-\;2\,\mathbf{r}\cos(\theta-t)\;+\;\mathbf{r}^2}\cdot f(\mathrm{e}^{\mathbf{i}t})\,\mathrm{d}t\,.$$

Our $\mathrm{I}_f()$ is well-defined for $\mathbf{r}\neq 1$. □

jk#**Companion is harmonic**[P]**.** If the above f is real-valued on U, prove that I_f of (3c) is harmonic.

3d: Lemma. *Suppose $f:\widehat{\mathsf{U}}\to\mathbb{C}$ is analytic. Prove that companion fnc I_f, (3c), equals f on open-ball $\mathring{\mathsf{U}}$.* ◊

jk#**Matching a boundary condition**[P]**.** For a cts $f{:}\mathbb{P}^\circ \to \mathbb{R}$, prove that companion fnc I_f of (3c) has radial limits equaling f.

Unfinished: as of 18Mar2024

FC#**4.32** P *68*. Suppose f and g are holomorphic in region G, and γ is a simple piecewise smooth G-contractible path. Prove: If $f{=}g$ on γ, then $f(z) = g(z)$ for *all* z lying *inside* γ.

Chapter 5

FC#**5.01**[P.] *79*. Let R be the $\pm[4 \pm 4\mathbf{i}]$ square, positively oriented . Compute $\mathbf{I} := \oint_{\mathsf{R}} \frac{\exp(z^2)}{z^3}\,\mathrm{d}z$. Compute $\mathbf{J} := \oint_{\mathsf{R}} \frac{\exp(z)\cos(z)}{[z - \boldsymbol{\pi}]^3}\,\mathrm{d}z$.

FC#**5.13**[P.] *79*. Suppose f is entire and $|f(z)| \leq \sqrt{|z|}$, for all $z \in \mathbb{C}$. Prove f is identically 0.

jk#**GINT**[P.]. Compute

$$J := \oint_{\mathsf{C}_7} \frac{\mathsf{e}^{3z}}{[z-2]^9}\,\mathrm{d}z\,.$$

Chapter 7

FC#7.23$^{P.}$*108*. Let $f_n(x) = n^2 x\, \mathsf{e}^{-nx}$.

a: Prove $\lim_{n\to\infty} f_n(x) = 0$, for all $x \geq 0$.

b: Find $\lim_{n\to\infty} \int_0^1 f_n$.

c: Why doesn't your answer to part (b) violate Prop 7.27 [about unif-convergence of fncs]?

FC#7.26$^{P.}$*108*. Find the power series, centered at the origin, of each of the following functions.

a: $\cos(z)$ b: $\cos(z^2)$ c: $z^2 \sin(z)$ d: $\left[\sin(z)\right]^2$.

Chapter 8

jk#**Laurent series**[P]. Compute the power-series $\sum_{n=0}^{\infty} B_n z^n$ for $F(z) := \dfrac{1}{z-3}$, on ball $\mathcal{B} := \mathrm{Bal}_3(0)$.

For $F()$, compute Laurent series $\sum_{n\in\mathbb{Z}}^{\infty} A_n z^n$ on annulus $\mathcal{A} := \mathrm{Ann}_{\infty}^{3}(0)$.

Chapter 9

FC#**9.21c**[P]*141*. On $\mathbf{A} := \{1 \le |z| \le 2\}$, a closed annulus, how many zeros does $f(z) := z^4 - 5z + 1$ have?

Misc

Ahl#2 $^{P.}$*123*. For a posint N and $R>0$, entire fnc f satisfies $|f(z)| \le |z|^N$, for each $z \overset{|\cdot|}{>} R$. Prove that $f(z)$ is a polynomial in z.

Ahl#3 $^{P.}$*120*. With $\mathsf{C} := \mathrm{Sph}_3(0)$, compute

$$J_{\mathbf{p}} := \int_{\mathsf{C}} \frac{|\mathrm{d}z|}{|z - \mathbf{p}|^2}$$

assuming that $\mathbf{p} \notin \mathsf{C}$.

Predictions. At the origin, $p = 0$, we certainly expect

$$J_0 \overset{\text{should}}{=\!=\!=} \frac{1}{3^2} \cdot 2\boldsymbol{\pi} \cdot 3 = \frac{6\boldsymbol{\pi}}{9}. \qquad [\textit{See } (*).]$$

Mapping $\mathbf{p} \mapsto J_{\mathbf{p}}$ is cts, and always positive. Finally,

$$\lim_{\mathbf{p} \to \boldsymbol{\infty}} J_{\mathbf{p}} = 0 \quad \textit{and} \quad \lim_{|\mathbf{p}| \to 3} J_{\mathbf{p}} = \infty. \qquad \square$$

Ahl#5$^{P.}$*123*. Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(\mathbf{p})| > n!{\cdot}n^n$. Formulate a sharper theorem of the same kind.

jk#Loopy$^{P.}$. Consider parametrization $\sigma\colon[0,2\pi]\to\mathbb{C}$ by

$$\sigma(t) \;:=\; [3+\cos(t)]\cdot \mathrm{e}^{\mathbf{i}\cdot 2t}$$

Compute $J \;:=\; \oint_\sigma \frac{\exp(z+4)}{z\,[z-3]}\,\mathrm{d}z$.

jk1: Prove $[Tz, Tq_0, Tq_1, Tq_\infty] = [z, q_0, q_1, q_\infty]$, for each cross-ratio and each LFT T.

jk2: For distinct points $\mathbf{b},\mathbf{c},\mathbf{d} \in \widehat{\mathbb{C}}$, let $\widehat{\mathbf{b},\mathbf{c},\mathbf{d}}$ mean the gencirc they determine. Prove for each $a \in \widehat{\mathbb{C}}$: *Point a lies in the gencirc IFF crossratio $[a, \mathbf{b}, \mathbf{c}, \mathbf{d}]$ is (extended-)real.*

§A *Appendix:* Möbius

Möbius transformation

With $\widehat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$ denoting the extended complex-plane (holomorphically equiv. to the Riemann Sphere) we define the ***Möbius group*** $(\mathbb{M}, \circ, Id_{\widehat{\mathbb{C}}})$; it is the automorphism group of $\widehat{\mathbb{C}}$. This $\mathbb{M}$ is the set functions defined by LFTs or, equivalently, by CrossRatios. (Both LFTs and CRs are defined below).

LFT. A ***linear fractional transformation*** is a map

4a: $$f(z) := \frac{\mathtt{a}z + \mathtt{b}}{\mathtt{c}z + \mathtt{d}}, \quad \text{where } \mathtt{a,b,c,d} \in \mathbb{C} \text{ and } \mathtt{ad} - \mathtt{bc} \neq 0.$$

We will associate $\frac{\mathtt{a}z+\mathtt{b}}{\mathtt{c}z+\mathtt{d}}$ with the 2×2 matrix $\left[\begin{smallmatrix}\mathtt{a} & \mathtt{b}\\ \mathtt{c} & \mathtt{d}\end{smallmatrix}\right]$, whose determinant is $\text{Det}(\left[\begin{smallmatrix}\mathtt{a} & \mathtt{b}\\ \mathtt{c} & \mathtt{d}\end{smallmatrix}\right]) = \mathtt{ad} - \mathtt{bc}$. If the LFT is understood, I may write Det to stand for the corresponding $\mathtt{ad} - \mathtt{bc}$ quantity.

LFT $f(z) = \frac{\mathtt{a}z+\mathtt{b}}{\mathtt{c}z+\mathtt{d}}$ is a map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$; from the extended plane to itself. Indeed,

$$f(\infty) := \lim_{z\to\infty} \frac{\mathtt{a}z+\mathtt{b}}{\mathtt{c}z+\mathtt{d}} = \frac{\mathtt{a}}{\mathtt{c}}. \quad \textit{And} \quad f(\tfrac{-\mathtt{d}}{\mathtt{c}}) = \infty.$$

N.B: In (4a), multiplying the four parameters by a non-zero complex does not change the defined function. E.g $\frac{5\mathtt{a}\cdot z + 5\mathtt{b}}{5\mathtt{c}\cdot z + 5\mathtt{d}}$ is the same $f(z)$ that (4a) defined. Consequently, (4a) defines a **3**-$\mathbb{C}$-dim'al group [i.e, *not* 4-$\mathbb{C}$-dim'al].

Normalizing. *When numbers* a,b,c,d *are real*: LFT $\frac{\mathtt{a}z+\mathtt{b}}{\mathtt{c}z+\mathtt{d}}$ is ***normalized*** if Det$=1$ *and*

$*$N: *If* $\mathtt{c}\neq 0$, *then* $\mathtt{c} > 0$; *if* $\mathtt{c}=0$, *then* $\mathtt{d} > 0$.

In contrast, the LFT is "$\mathbb{Z}$-***normalized***" if a,b,c,d are *integers* and $\text{GCD}(\mathtt{a,b,c,d}) = 1$, *and* ($*$N) holds. So the $\mathbb{Z}$-normalized presentation of $\frac{65z+20}{-10z+15}$ is $\frac{-13z-4}{2z-3}$ i.e, $\frac{-13z + -4}{2z + -3}$.

And $\mathbb{Z}$-normalizing $\frac{z-4}{-3/2}$ yields $\frac{-2z+8}{3}$.

Composition of LFTs. Consider LFTs

$$g(z) := \frac{\alpha z + \beta}{\gamma z + \delta} \quad \textit{and} \quad f(z) := \frac{\mathtt{a}z + \mathtt{b}}{\mathtt{c}z + \mathtt{d}}.$$

One checks easily that their composition $g \circ f$ is the LFT whose matrix is the matrix-product

$*$: $$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}\mathtt{a} & \mathtt{b}\\ \mathtt{c} & \mathtt{d}\end{bmatrix} \overset{\text{note}}{=\!=} \begin{bmatrix}\alpha\mathtt{a}+\beta\mathtt{c} & \alpha\mathtt{b}+\beta\mathtt{d}\\ \gamma\mathtt{a}+\delta\mathtt{c} & \gamma\mathtt{b}+\delta\mathtt{d}\end{bmatrix}.$$

In general, LFT $f \circ g$ differs from $g \circ f$; unsurprizing, as matrix-mult is not commutative. One checks that the product of the determinants of the matrices on LhS($*$) equals Det(RhS($*$)).

When the Det of $\mathsf{M} := \left[\begin{smallmatrix}\mathtt{a} & \mathtt{b}\\ \mathtt{c} & \mathtt{d}\end{smallmatrix}\right]$ is non-zero, then

$$\mathsf{M}^{-1} = \frac{1}{\text{Det}(\mathsf{M})} \cdot \begin{bmatrix}\mathtt{d} & -\mathtt{b}\\ -\mathtt{c} & \mathtt{a}\end{bmatrix}.$$

(Recall that multiplying a matrix by a scalar s simply multiplies each entry by s. E.g $5 \cdot \left[\begin{smallmatrix}\mathtt{a} & \mathtt{b}\\ \mathtt{c} & \mathtt{d}\end{smallmatrix}\right] = \left[\begin{smallmatrix}5\mathtt{a} & 5\mathtt{b}\\ 5\mathtt{c} & 5\mathtt{d}\end{smallmatrix}\right]$.) So the inverse-fnc of the f of (4a) can be written as

4b: $$f^{-1}(z) = \frac{\mathtt{d}z - \mathtt{b}}{-\mathtt{c}z + \mathtt{a}}.$$

Generating $\mathbb{M}$. Especially simple are the *Translation*, *Dilation*, and *Inversion/Reciprocation* LFTs:

$$\begin{aligned} \mathsf{T}_\tau &:= z \mapsto z+\tau; \\ \mathsf{D}_m &:= z \mapsto mz; \\ \mathsf{R} &:= z \mapsto 1/z, \end{aligned}$$

where $\tau, m \in \mathbb{C}$ with $m \neq 0$. An arbitrary $f(z) := \frac{\mathtt{a}z+\mathtt{b}}{\mathtt{c}z+\mathtt{d}}$ can be built from these, as follows.

CASE: c=0 LFT is $\frac{\mathtt{a}}{\mathtt{d}}z + \frac{\mathtt{b}}{\mathtt{d}}$; so $f = \mathsf{T}_{\frac{\mathtt{b}}{\mathtt{d}}} \circ \mathsf{D}_{\frac{\mathtt{a}}{\mathtt{d}}}$.

CASE: c≠0 *Normalize* the LFT by $\mathtt{ad} - \mathtt{bc} = 1$; so $\mathtt{ad} - 1 \overset{\dagger:}{=} \mathtt{bc}$. We claim

‡: $$f = \mathsf{T}_{\mathtt{a}/\mathtt{c}} \circ \mathsf{D}_{-1/\mathtt{c}^2} \circ \mathsf{R} \circ \mathsf{T}_{\mathtt{d}/\mathtt{c}}.$$

Computing, $\text{RhS}_\ddagger(z) = \frac{\mathtt{a}}{\mathtt{c}} + \frac{-1}{\mathtt{c}^2} \cdot \frac{1}{z + \frac{\mathtt{d}}{\mathtt{c}}}$. So $\mathtt{c} \cdot \text{RhS}_\ddagger(a)$ is

$$\mathtt{a} + \frac{-1}{\mathtt{c}z + \mathtt{d}} = \frac{\mathtt{ac}z + \mathtt{ad} - 1}{\mathtt{c}z + \mathtt{d}} \overset{\text{by }(\dagger)}{=\!=} \frac{\mathtt{ac}z + \mathtt{bc}}{\mathtt{c}z + \mathtt{d}},$$

which indeed equals $\mathtt{c} \cdot \frac{\mathtt{a}z+\mathtt{b}}{\mathtt{c}z+\mathtt{d}}$.

Möbius group as quotient. The set of 2×2 matrices with *non-zero determinant* has the anodyne moniker of ***General Linear group***. When the entries come from $\mathbb{C}$, it is written $\mathrm{GL}_{2\times 2}(\mathbb{C})$.

Since multiplying a 2×2 matrix by a non-zero constant does not change the LFT that the matrix determines, our Möbius group is the quotient

$$\mathrm{GL}_{2\times 2}(\mathbb{C}) \Big/ \overset{\times\,\mathrm{nzc}}{\frown} \;;$$

the set of equivalence classes.

Cross ratio. [... is not an angry ratio] is an alternative way of defining the Möbius group.

$$5:\quad [z,\,q_0,\,q_1,\,q_\infty] := \frac{[z-q_0][q_1-q_\infty]}{[z-q_\infty][q_1-q_0]},\quad \textit{with } q_0,q_1,q_\infty \textit{ distinct in } \widehat{\mathbb{C}}.$$

When one of q_0, q_1, q_∞ is ∞, we then interpret the CR as a limit:

$$[z,\,\infty,\,q_1,\,q_\infty] := \lim_{v\to\infty}[z,\,v,\,q_1,\,q_\infty] = \frac{0z+[q_1-q_\infty]}{z-q_\infty};$$

$$\dagger:\quad [z,\,q_0,\,\infty,\,q_\infty] := \lim_{v\to\infty}[z,\,q_0,\,v,\,q_\infty] = \frac{z-q_0}{z-q_\infty};$$

$$[z,\,q_0,\,q_1,\,\infty] := \lim_{v\to\infty}[z,\,q_0,\,q_1,\,v] = \frac{z-q_0}{0z+[q_1-q_0]}.$$

With $f(z) := [z,\,q_0,\,q_1,\,q_\infty]$, note that

$$f(q_0) = 0,\quad f(q_1) = 1,\quad f(q_\infty) = \infty.$$

Etymology. In German, a cross-ratio was called a *Doppelverhältnis* [double ratio] because ...

$$(5)':\quad [z,\,q_0,\,q_1,\,q_\infty] \stackrel{\text{note}}{=\!=} \frac{z-q_0}{q_1-q_0}\Big/\frac{z-q_\infty}{q_1-q_\infty},$$

... it is a *ratio of ratios.* □

Crossratio ↔ LFT. As fncs of $z\in\widehat{\mathbb{C}}$, suppose we have equality

$$*:\quad \frac{\mathtt{a}z+\mathtt{b}}{\mathtt{c}z+\mathtt{d}} = [z,\,q_0,\,q_1,\,q_\infty].$$

Computing $\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}$ ***from the CR.*** When one of q_0,q_1,q_∞ is ∞, our (†) gives

$$\begin{matrix}\mathtt{a}=0, & \mathtt{b}=q_1-q_\infty\\ \mathtt{c}=1, & \mathtt{d}=-q_\infty\end{matrix};$$

$$\begin{matrix}\mathtt{a}=1, & \mathtt{b}=-q_0\\ \mathtt{c}=1, & \mathtt{d}=-q_\infty\end{matrix};$$

$$\begin{matrix}\mathtt{a}=1, & \mathtt{b}=-q_0\\ \mathtt{c}=0, & \mathtt{d}=q_1-q_0\end{matrix}.$$

Otherwise, when *none* of q_0,q_1,q_∞ is ∞, use this:

$$\ddagger:\quad \begin{matrix}\mathtt{a} = q_1-q_\infty & \textit{and} & \mathtt{b} = [q_\infty-q_1]\cdot q_0,\\ \mathtt{c} = q_1-q_0 & \textit{and} & \mathtt{d} = [q_0-q_1]\cdot q_\infty.\end{matrix}$$

Computing q_0,q_1,q_∞ ***from the LFT.*** Voila:

$$\ddagger\ddagger:\quad \begin{aligned} q_0 &= -\mathtt{b}/\mathtt{a};\\ q_1 &= [\mathtt{d}-\mathtt{b}]/[\mathtt{a}-\mathtt{c}];\\ q_\infty &= -\mathtt{d}/\mathtt{c}.\end{aligned}$$

As usual, if a denominator is zero, interpret the formulas by taking a limit. E.g, if $\mathtt{a}=0$ then $\mathtt{b}\neq 0$ since $\text{Det}\neq 0$. Thus $q_0 = \frac{-\mathtt{b}}{0} = \infty$. As expected, the point that $f(z) \stackrel{\text{note}}{=\!=} \frac{b}{cz+d}$ maps to 0 is indeed $z{=}\infty$.

Inverse-fnc of crossratio. Distinct points $r_0,r_1,r_\infty\in\mathbb{C}$ engender $w = f(z) := [z,\,r_0,\,r_1,\,r_\infty]$. We seek points $q_0,q_1,q_\infty\in\widehat{\mathbb{C}}$ so that

$$z = f^{-1}(w) := [w,\,q_0,\,q_1,\,q_\infty].$$

A matrix for f is $\left[\begin{smallmatrix}\mathtt{a}&\mathtt{b}\\ \mathtt{c}&\mathtt{d}\end{smallmatrix}\right]$ from (‡). Hence, $\underline{a}$ matrix for f^{-1} is $\left[\begin{smallmatrix}A&B\\ C&D\end{smallmatrix}\right] := \left[\begin{smallmatrix}\mathtt{d}&-\mathtt{b}\\ -\mathtt{c}&\mathtt{a}\end{smallmatrix}\right]$. So (‡‡) and (‡) give

$$¥:\quad \begin{aligned} q_0 &= \frac{-B}{A} = \frac{\mathtt{b}}{\mathtt{d}} \stackrel{\text{note}}{=\!=} \frac{[r_1-r_\infty]\cdot r_0}{[r_1-r_0]\cdot r_\infty} \stackrel{\text{note}}{=\!=} q_\infty\cdot\frac{r_0}{r_\infty};\\ q_1 &= \frac{D-B}{A-C}\\ &= \frac{\mathtt{a}+\mathtt{b}}{\mathtt{d}+\mathtt{c}} = \frac{[r_1-r_\infty][1-r_0]}{[r_1-r_0][1-r_\infty]} \stackrel{\text{note}}{=\!=} q_\infty\cdot\frac{1-r_0}{1-r_\infty};\\ q_\infty &= \frac{-D}{C} \stackrel{\text{note}}{=\!=} \frac{\mathtt{a}}{\mathtt{c}} = \frac{r_1-r_\infty}{r_1-r_0}.\end{aligned}$$