Ord: ___ ___

Sets and Logic
MHF3202 7860

Class-C

Prof. JLF King
Wedn, 10Apr2024

C1: Short answer. Show no work. Write LARGE.

Write **DNE** if the object does not exist or the operation cannot be performed. NB: DNE $\neq \{\} \neq 0$.

20 a EoS 2024 *Games Party*, from 12:50pm–4:30pm, will take place at *Pascal's Cafe* on Wedn, 24Apr.

Circle **Yes!** **True!** **I'll-bring-a-game!**

10 10 b Relation $\mathbf{R}$ is a binrel on set $\mathbb{Z}_+$, defined by $x\mathbf{R}y$ IFF $x^2 = 5y$.

Assertion "*Relation* $\mathbf{R}$ *is reflexive*" is T F

Assertion "*Relation* $\mathbf{R}$ *is antireflexive*" is T F

Soln: As $\neg[1\mathbf{R}1]$, since $1^2 \neq 5 \cdot 1$, our $\mathbf{R}$ is *not* refl.
Note $5\mathbf{R}5$, since $5^2 = 5 \cdot 5$, hence $\mathbf{R}$ is *not* anti-refl.

C 25 Note that $\mathrm{GCD}(55,33,15)=1$. Find particular integers S,T,U so that $55S+33T+15U=1$:

$S=4$, $T=-8$, $U=3$.

[*Hint:* $\mathrm{GCD}\Big(\mathrm{GCD}(55,33),\ 15\Big)=1$.]

Soln. Among the ∞ly many correct triples are $(7,-8,-8)$, $(4,-8,3)$, $(1,-3,3)$, $(-2,2,3)$, $(-8,12,3)$ and $(-23,7,69)$.

LBolting,

```
(lightning 55 33)

  n:   r_n   q_n   s_n   t_n
/---------------------------\
  0:    55    --     1     0
  1:    33     1     0     1
  2:    22     1     1    -1
  3:    11     2    -1     2
  4:     0 Infty     3    -5
\___________________________/

  Rename r3,s3,t3  to  GCD,S,T.

  Note    GCD   =  r0 * S  +  r1 * T,   i.e
           11   =  [55]*[-1] + [33]*[2].
```

And, we now want `(lightning 11 15)` but I'll reverse the order to make the table one row shorter.

```
(lightning 15 11)

  n:   r_n   q_n   s_n   t_n
/---------------------------\
  0:    15    --     1     0
  1:    11     1     0     1
  2:     4     2     1    -1
  3:     3     1    -2     3
  4:     1     3     3    -4
  5:     0 Infty   -11    15
\___________________________/
```

So $1 = 11\cdot[-4] + 15\cdot 3$. Substituting from the 1st LBolt,

$$\begin{aligned} 1 &= \Big[55\cdot[-1] + 33\cdot 2\Big]\cdot[-4] + 15\cdot 3 \\ &= 55\cdot 4 + 33\cdot[-8] + 15\cdot 3\,. \end{aligned}$$ ♦

Exploration. Since $\mathrm{GCD}(55,33)=11$, the 1-parameter family of Bézout-pairs for $(55,33)$ is

$$\begin{aligned} 11 &= \Big[-1 + \tfrac{33}{11}\cdot n\Big]\cdot 55 + \Big[2 - \tfrac{55}{11}\cdot n\Big]\cdot 33 \\ &= \underbrace{\Big[-1 + 3n\Big]}_{A_n}\cdot 55 + \underbrace{\Big[2 - 5n\Big]}_{B_n}\cdot 33\,. \end{aligned}$$

Our second LBolt gives that

$$1 = [-4]\cdot 11 + 3\cdot 15\,.$$

Our 1-parameter family of $(11,15)$-Bézout-pairs is thus

$$1 = \underbrace{[-4 + 15k]}_{X_k}\cdot 11 + \underbrace{[3 - 11k]}_{Y_k}\cdot 15\,.$$

Rewriting, $1 = 11\,X_k + 15\,Y_k$. Substituting,

$$\begin{aligned} 1 &= \Big[55A_n + 33B_n\Big]\cdot X_k + 15\cdot Y_k \\ &= 55\cdot\underbrace{A_n X_k}_{P_{n,k}} + 33\cdot\underbrace{B_n X_k}_{Q_{n,k}} + 15\cdot\underbrace{Y_k}_{R_{n,k}}\,. \end{aligned}$$

An *incomplete* 2-parameter family $\mathbf{V}_{n,k} := \begin{bmatrix} P_{n,k} \\ Q_{n,k} \\ R_{n,k} \end{bmatrix}$ of Bézout-triples for $(55,33,15)$ can be cheerfully written as

$$\dagger:\quad \begin{aligned} P_{n,k} &= \Big[-1 + 3n\Big]\cdot[-4 + 15k]\,, \\ Q_{n,k} &= \Big[2 - 5n\Big]\cdot[-4 + 15k] \quad \textit{and} \\ R_{n,k} &= [3 - 11k]\,. \end{aligned}$$

Plugging in $n=0$ and $k=0$ gives the above $\begin{bmatrix} 4 \\ -8 \\ 3 \end{bmatrix}$. ♦

Going further. While correct triples $\begin{bmatrix} 7 \\ -8 \\ -8 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$. can be obtained from (†), we need *non-integer* values of n,k to get them; e.g $\begin{bmatrix} 7 \\ -8 \\ -8 \end{bmatrix} = \mathbf{V}_{\frac{30}{55},1}$. *Unpleasant...*

Using another method [*Smith normal form* of a matrix], one can derive

$$\ddagger:\quad \mathbf{U}_{\alpha,\beta} := \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} + \alpha\begin{bmatrix} -3 \\ 10 \\ -11 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ -5 \\ 11 \end{bmatrix}$$

which parametrizes all Bézout-triples, as α,β range over all the integers. E.g $\mathbf{U}_{-2,-3} = \begin{bmatrix} 7 \\ -8 \\ -8 \end{bmatrix}$ and $\mathbf{U}_{1,1} = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$. ♦

d 15 On a 4-set, let K be the number of partitions which use *only* atom sizes 1 and 2. [A partition need not use both sizes.] Then $K=$ ________. [Multinomial coefficients are useful.]

Counting ptns. Below, let $\#P$ mean "number of partitions with...". Let a symbol such as $4A1$ abbreviate "4 atoms of size 1".

Computing,

$$K = \frac{\overbrace{\binom{4}{1,1,1,1}}^{\#P\ 4A1}}{4!} + \frac{\overbrace{\binom{4}{1,1,2}}^{\#P\ 2A1\ 1A2}}{2!} + \frac{\overbrace{\binom{4}{2,2}}^{\#P\ 2A2}}{2!}$$
$$= 1 + 6 + 3 = 10.$$

Alternatively,

$$K = 15 - [\#P\ 1A1\ 1A3] - [\#P\ 1A4]$$
$$= 15 - 4 - 1 = 10.$$

♦

e 10 10 The number of permutations of "PREPPER", as a multinomial coefficient, is $\binom{7}{3,2,2} = \binom{7}{3}\binom{4}{2}\binom{2}{2} = 35 \cdot 6 \cdot 1 \overset{\text{numeral}}{=\!=} 210$.

Counting perms. Label pockets 'P', 'R', 'E'. Tokens placed in the pockets are letter-positions 1,2,3,4,5,6,7. Hence $\binom{7}{3,2,2}$ is the number of permutations of "PREPPER". ♦

Note that I did not write: "The answer is $\binom{7}{3,2,2}$."

C2: Short answer. Show no work. Write LARGE.

10 10 f An explicit bijection $F:\mathbb{N}\hookrightarrow\!\!\!\rightarrow\mathbb{Z}$ is this:

When n is *even*, then $F(n) :=$ $n\,/\,2$.

When n is *odd*, then $F(n) :=$ $-[n+1]/2$.

∃ *other correct formulae.* A cute formula is

$$F(_{\text{even}}\, n) \;:=\; [-1]^{\frac{n}{2}} \cdot \frac{n}{2} \quad ;$$

$$F(_{\text{odd}}\, n) \;:=\; [-1]^{\frac{n-1}{2}} \cdot \frac{n+1}{2}\,. \quad ♦$$

ADDENDUM: Just for fun, a Computer Run:

```
(defun NegOneTo (e) (if (oddp e) -1 +1))

(defun fe (n) (* (NegOneTo (/ n 2)) (/ n 2)))
(defun fd (n) (* (NegOneTo (/ (1- n) 2)) (/ (1+ n) 2)))

(iter (for n :below 23) (format t " ~2D" (if (oddp n) (fd n) (fe n))))
  0  1 -1 -2  2  3 -3 -4  4  5 -5 -6  6  7 -7 -8  8  9 -9 -10 10 11 -11
```

g 35 Let $\mathbb{S}_{\mathbb{N}}$ be the set of *permutations of* $\mathbb{N}$. Circle those of following sets which are equinumerous with $\mathbb{N}^{\mathbb{N}}$:

$\mathbb{N}$ $\quad$ ($\mathbb{R}$) $\quad$ ($\mathbb{N}\times\mathbb{R}$) $\quad$ $2^{\mathbb{R}}$ $\quad$ ($\mathbb{R}^{\mathbb{N}}$) $\quad$ $\mathbb{R}^{\mathbb{R}}$ $\quad$ ($\mathbb{S}_{\mathbb{N}}$)

[Schröder-Bernstein is useful for some of these.]

Equi-card: [Below, a blue set is one of the candidate answers in the question.]

Our "Primer on Cardinalities" gives $2 \preccurlyeq \mathbb{N} \preccurlyeq \mathbb{R}$, and consequently

$$2^{\mathbb{N}} \preccurlyeq \mathbb{N}^{\mathbb{N}} \preccurlyeq \mathbb{R}^{\mathbb{N}} \asymp [2^{\mathbb{N}}]^{\mathbb{N}} \overset{\text{CE}}{\asymp} 2^{\mathbb{N}\times\mathbb{N}} \asymp 2^{\mathbb{N}}.$$

So S-B says $2^{\mathbb{N}} \asymp \mathbb{N}^{\mathbb{N}} \asymp \mathbb{R}^{\mathbb{R}}$. Recall, $2^{\mathbb{N}} \asymp \mathbb{R} \asymp \mathbb{N}\times\mathbb{R}$.

What about $\mathbb{S}_{\mathbb{N}}$*?* Examining the set-of-permutations, the identity map injects $\mathbb{S}_{\mathbb{N}} \overset{Id}{\hookrightarrow} \mathbb{N}^{\mathbb{N}} \asymp 2^{\mathbb{N}}$.

For a reverse injection, write $2^{\mathbb{N}}$ as $\{\mathsf{Stay}, \mathsf{Flip}\}^{\mathbb{N}}$. Inject $\{\mathsf{Stay}, \mathsf{Flip}\}^{\mathbb{N}} \hookrightarrow \mathbb{S}_{\mathbb{N}}$ by mapping $g \mapsto \widehat{g}$, where:

This $\widehat{g}$ *exchanges* $2n$ *with* $2n+1$ *IFF* $g(n) = \mathsf{Flip}$.

Specifically: *For* $n = 0, 1, 2, \ldots$

If $g(n) = \mathsf{Stay}$ then $\widehat{g}(2n) := 2n$ *and* $\widehat{g}(2n+1) := 2n+1$;
If $g(n) = \mathsf{Flip}$ then $\widehat{g}(2n) := 2n+1$ *and* $\widehat{g}(2n+1) := 2n$.

Now Schröder-Bernstein says $\mathbb{S}_{\mathbb{N}} \asymp 2^{\mathbb{N}}$.

NOT Equi: Cantor diagonalization showed $\mathbb{N} \prec 2^{\mathbb{N}}$, so $\mathbb{N}$ is too small.

OTOHand, $2^{\mathbb{R}}$ is too big, as $2^{\mathbb{R}} \succ \mathbb{R} \asymp 2^{\mathbb{N}}$. Even ?bigger?, $\mathbb{R}^{\mathbb{R}} \succcurlyeq 2^{\mathbb{R}}$. [Exer: Actually, $\mathbb{R}^{\mathbb{R}} \asymp 2^{\mathbb{R}}$.]

15 15 h Each three sets Ω,B,C engender a natural bijection, $\Theta\colon\Omega^{B\times C}\hookrightarrow\!\!\!\!\rightarrow[\Omega^B]^C$, defined, for each $f\in\Omega^{B\times C}$, by

$$\Theta(f) := \Big[c\mapsto \big[\, b\mapsto f\big((b,c)\big) \quad\ldots\ldots\ldots\ldots\ldots\ldots \big]\Big].$$

Its inverse-map $\Upsilon\colon[\Omega^B]^C\hookrightarrow\!\!\!\!\rightarrow\Omega^{B\times C}$ has, for $g\in[\Omega^B]^C$,

$$\Upsilon(g) := \Big[(b,c)\mapsto \big[\, [g(c)](b) \quad\ldots\ldots\ldots\ldots\ldots\ldots \big]\Big].$$

Curious?? See *Currying a function*:

https://en.wikipedia.org/wiki/Currying

and

https://en.wikipedia.org/wiki/Currying#Set_theory i 30 A "Cantor's-Hotel" type bijection $f\colon (5,6] \hookrightarrow\!\!\!\rightarrow (0,1)$ is:

$f(\ \frac{1}{n} + 5\ ) :=\ \frac{1}{n+1}$, for *each* posint n;

and $f(x) :=\ x - 5$, for *each* $x \in (5,6] \setminus C$,

where $C := \left\{ \frac{1}{n} + 5 \;\middle|\; n \in \mathbb{Z}_+ \right\}$.

C1: ___ ___ ___ 100pts

C2: ___ ___ ___ 115pts

Total: ___ ___ ___ 215pts

NAME: Energetic *Proof-is-my-Middle-Name* Student

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: *Energetic Student*