Burnside's lemma applied to necklaces

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Group action. The symbol $G \circlearrowleft \Omega$ means that gp G ***acts on*** set Ω; here, that $\boxed{G \subset \mathbb{S}_\Omega}$. For $h \in G$ and $\omega \in \Omega$, write the gp-action as $h(\omega)$ or just $h\omega$.

Recall: The ***orbit*** and ***stabilizer*** of a point ω, and the ***fixed-pt set*** of a group-element h, are:

$$\begin{aligned}
\mathcal{O}(\omega) \;&=\; \mathcal{O}_G(\omega) \;\coloneqq\; \{h(\omega) \;\;|\;\; h \in G\,\} && \subset\; \Omega\,;\\
\mathrm{Stab}(\omega) \;&=\; \mathrm{Stab}_G(\omega) \;\coloneqq\; \{h \in G \mid h(\omega) = \omega\} && \subset\; G\,;\\
\mathrm{Fix}(h) \;&=\; \mathrm{Fix}_G(h) \;\coloneqq\; \{\omega \in \Omega \mid h(\omega) = \omega\} && \subset\; \Omega\,.
\end{aligned}$$

Counting necklaces

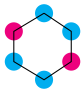
We'll use the following algebraic tool.

1: Burnside's Lemma. *The number of G-orbits is*

‡:
$$^{\#}\textit{Orbits} \;=\; \frac{1}{|G|}\cdot\sum_{g\in G}\big|\mathrm{Fix}(g)\big|\,. \qquad \Diamond$$

Henceforth, an N***-necklace*** is an oriented circular ring of N colored stones. Two necklaces are the same if one can be rotated [but *not* turned-over] to get the other necklace. Hence rotation group $G \coloneqq (\mathbb{Z}_N, +, 0)$ acts on the space, Ω, of necklaces. Use symbol $\mathtt{R}$ (for Rotation) for a generator of G.

2: Necklace-6. *With two colors of stone, how many 6-stone necklaces are there?*◊

Soln. A necklace *might* be periodic with period d, where d is a divisor of 6. E.g, necklace  is invariant under rotation by $d{=}3$ stone-positions. The number of necklaces invariant under rotation by d is 2^d, since each of the d positions has a choice of 2 colors.

In $\mathbb{Z}_6$, the *multiples* of d form a subgp isomorphic to $\mathbb{Z}_\ell$, where $d{\cdot}\ell = 6$; this cyclic subgp has $\varphi(\ell)$ many generators.

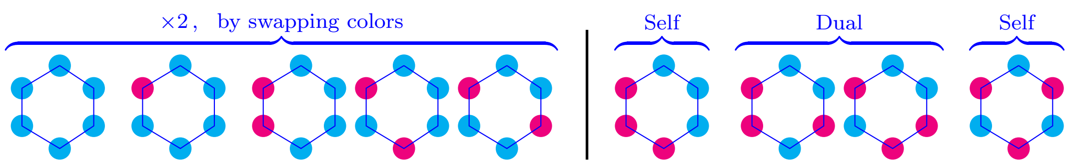
Burnside's Lemma can be applied systematically, using the table below:

Period d	$\ell := \frac{6}{d}$	$\varphi(\ell)$	#Necklaces fixed by $\mathtt{R}^d$
6	1	1	2^6
3	2	1	2^3
2	3	2	2^2
1	6	2	2^1

Burnside's says that #Necklaces equals

$$\frac{1}{|G|} \cdot \sum_{g \in G} \left|\mathrm{Fix}(g)\right| = \frac{1}{6}\left[1{\cdot}2^6 + 1{\cdot}2^3 + 2{\cdot}2^2 + 2{\cdot}2^1\right] = 14.$$

Here are all the 6-stone Aqua-Red necklaces:



Conveniently, $[5{\times}2] + 4$ indeed equals 14. ♦

General necklaces. Allowing K colors gives $\frac{1}{6}\left[K^6 + K^3 + 2{\cdot}K^2 + 2{\cdot}K\right]$ many 6-stone necklaces. Three colors makes 130 necklaces; four colors, 700. [*Exer: Show that polynomial* $K^6 + K^3 + 2K^2 + 2K$ *is always divisible by* 6, *for each natnum* K.]

Extending, the # of N-stone necklaces with K colors allowed, $\mathcal{L}_{N,K}$, is

3: $$\mathcal{L}_{N,K} = \frac{1}{N} \sum_{d\cdot\ell = N} \left[\varphi(\ell) \cdot K^d\right],$$

with the sum taken over all *ordered pairs* (d, ℓ) of posints satisfying $d{\cdot}\ell = N$.

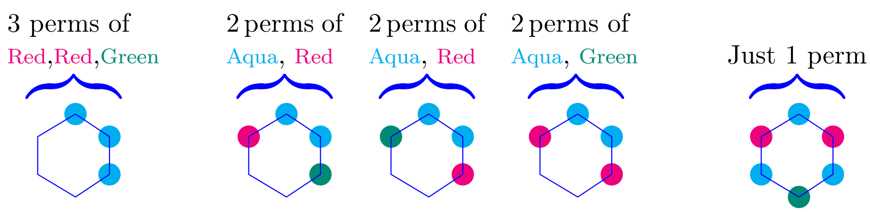
4: Specified color-counts. *How many 6-stone necklaces have: Three aqua pearls, two red pearls, and a single green pearl?* ♢

Pf. Let's make a table-of-values for $(3,2,1)$-necklaces:

Period d	$\ell := \frac{6}{d}$	$\varphi(\ell)$	#Necklaces fixed by $\mathtt{R}^d$
$\lceil 1^6 \rfloor, 6$	1	1	$\binom{6}{3,2,1}$
$\lceil 2^3 \rfloor, 3$	2	1	0
$\lceil 3^2 \rfloor, 2$	3	2	0
$\lceil 6^1 \rfloor, 1$	6	2	0

The *Green*-stone must appear *exactly* once. Thus $\ell = \frac{6}{d}$ must be 1, so only the top row of the above table can have any fixed-pts.

Note $\binom{6}{3,2,1} = \binom{6}{3}\binom{3}{2}\binom{1}{1} = [5{\cdot}4] \cdot 3$. Hence the number of $(3,2,1)$-necklaces is $\frac{1}{6}\big[5{\cdot}4 \cdot 3\big] \overset{\text{note}}{=\!=} 10$. And *here they are:*



Happily, $3+2+2+2+1$ equals 10. ♦

5: Counting 9-bracelets. *How many 9-stone bracelets have 4 aqua stones, 3 red stones, 2 green stones?* [*A* ***bracelet*** *allows the ring-of-stones to be turned-over.*] ♢

Pf. The acting gp is $G := \mathbb{D}_9 = \langle \mathtt{R}, \mathtt{F} \rangle$. The following table counts #Fixed-pts:

Cyc-signature & example	The number of such elts	#Bracelets fixed by $\mathtt{R}^d$
$\lceil 1^9 \rfloor$, $\mathtt{R}^9$	$\varphi(\frac{9}{9}) = 1$	$\binom{9}{4,3,2}$
$\lceil 3^3 \rfloor$, $\mathtt{R}^3$	$\varphi(\frac{9}{3}) = 2$	0
$\lceil 9^1 \rfloor$, $\mathtt{R}^1$	$\varphi(\frac{9}{1}) = 6$	0
$\lceil 2^9 \rfloor$, $\mathtt{F}$	9	$\binom{4}{2,1,1}$

[Checking: $1+2+6+9$ equals $|\mathbb{D}_9|$.] The #stones-spectrum is $(4,3,2)$ but some values are not divisible by 3; so 0 bracelets are fixed under the $\lceil 3^3 \rfloor$ signature. Similarly, not all of $(4,3,2)$ is divisible by 9.

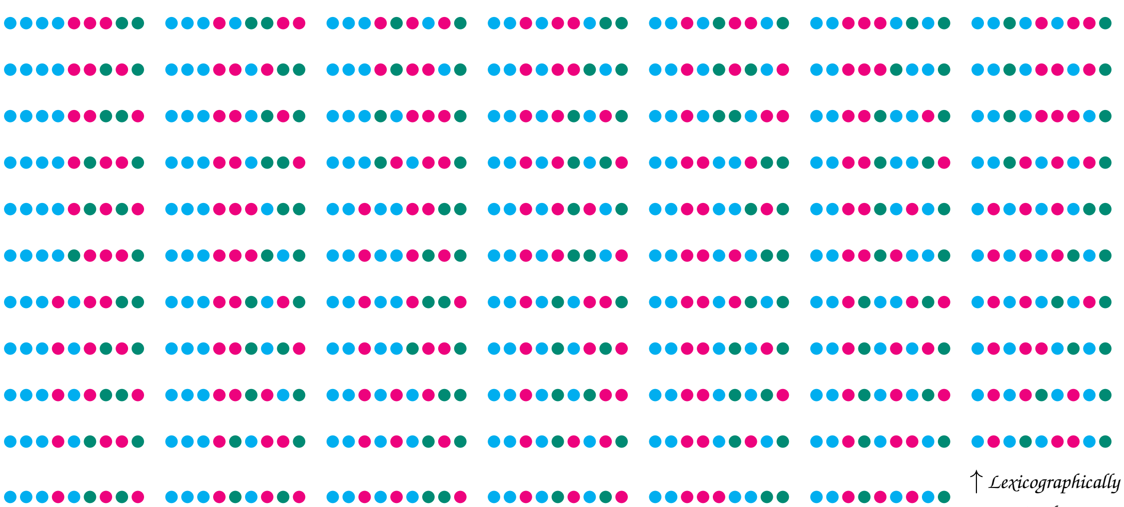
The only color with an odd-# of stones is red. So a flip can only fix a bracelet if the flip-axis passes through a red, leaving two 4-stone halves: *Each half with* $\frac{4}{2}$ *aqua stones*, $\frac{2}{2}$ *red stones*, $\frac{2}{2}$ *green stones*. So the # of flip-fixed bracelets is $\binom{4}{2,1,1} = 6 \cdot 2$.

Computing, $\binom{9}{4,3,2} = \binom{9}{4}\binom{5}{3} = 9{\cdot}7{\cdot}2 \cdot 10$. Consequently,

$$
\begin{aligned}
{}^{\#}\text{Bracelets} &= \frac{1}{|\mathbb{D}_9|}\left[1{\cdot}\binom{9}{4,3,2} + 9\cdot\binom{4}{2,1,1}\right] \\
&= \frac{1}{2{\cdot}9}\left[9{\cdot}7{\cdot}2\cdot 10 + 9{\cdot}6{\cdot}2\right] \\
&= 7\cdot 10 + 6 = 76\,.
\end{aligned}
$$

♦

The 76 bracelets:



↑ *Lexicographically maximal.*