Complex Analysis   Exam Beta   Prof. J. King
1999 March 29

Notation. **"nbhd"**, neighborhood. $C_r(w)$, the circle of radius r, centered at w. A ***region*** is a connected open non-void subset of the complexes. Use $\Re()$ and $\Im()$ for the real-part and imaginary-part operators.

[Problem $\beta 2$ is 170 points. The others are each 120 points.]

β**1:** Please prove this theorem from scratch.

SHRINKING-CIRCLE THM. *Suppose f is defined and continuous in a neighborhood of a point $w \in \mathbb{C}$. For small $r > 0$ let*

$$I_r := \int_{C_r(w)} \frac{f(z)}{z - w}\, dz\,.$$

Then $\lim_{r \searrow 0} I_r$ exists, and equals $2\pi\imath \cdot f(w)$.

β**2:** Please prove:

CAUCHY-GOURSAT THM. *Suppose $R \subset \mathbb{C}$ is a closed square* [with sides parallel to the axes], *and let $\Gamma := \partial R$. Suppose that h is analytic on R. Prove that $\int_\Gamma h(z)\, dz$ is zero.*

ISTFix $\varepsilon > 0$ and establish this:

$$\left| \int_\Gamma h(z)\, dz \right| \leq 100\varepsilon \cdot \operatorname{area}(R)\,.$$

What is the best value of "100" that you can obtain?

You may use without proof this fact: [*] *If g is a polynomial, then its integral around a square is zero.*

Explicitly tell me where you needed that h be differentiable *on* Γ.

β**3:** Please prove this:

MODULUS-CONSTANCY LEMMA. *Suppose h is analytic on an open set Λ. If $|h|$ is constant on Λ, then h is constant on Λ.*

β**4:** Carefully state the **Maximum-Modulus Thm**. Now use it to prove: *If f is analytic on a region Ω, and $\Re \circ f$ attains a maximum, or a minimum, on Ω, then f is constant on Ω.*

β**5:** Let C denote the circle of radius 3, centered at $P := 2 - 2\imath$. Let

$$g(z) := \frac{1}{[z^2 - 9] \cdot [z + \imath]^4}.$$

Please compute

$$\int_C g(z)\, dz\,.$$

You may express your answer in the form shown on the blackboard.

Make sure to cite the theorem(s) which permit your method of computation. Verify that your application actually satisfies the hypotheses of your theorem(s).

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