Staple!

Ord: ___ ___

Sets and Logic
MHF3202 7860

Class-B

Prof. JLF King
Friday, 22Mar2024 5 10 10

B4: Short answer. Show no work. Write LARGE.

Write **DNE** if the object does not exist or the operation cannot be performed. NB: DNE $\neq \{\} \neq 0$.

a 15 For a LOR (letter-of-recommendation), Prof. K requires two courses, or a Special Topics [e.g, my NUMBER THEORY AND CRYPTOGRAPHY], or graduate course Circle:

Yes **True** **Darn tootin'!**

b The ***Threeish-numbers*** comprise $\mathcal{T} := 1 + 3\mathbb{N}$. $\mathcal{T}$-number $385 \stackrel{\text{note}}{=\!=} 35 \cdot 11$ is $\mathcal{T}$-*irreducible*: T (F)

Irr Soln: *False*; $35 = 7 \cdot 5$. So $385 = 7 \cdot [5 \cdot 11]$ is a non-trivial *Threeish*-factorization of 385.

Threeish $N := 85$ is **not** $\mathcal{T}$-*prime* because $\mathcal{T}$-numbers $J :=$ .... and $K :=$ .... satisfy that $N \bullet\!| [J \cdot K]$, **yet** $N \nmid J$ and $N \nmid K$.

Prime Solution: Say that an integer k is ***3Neg*** if $k \equiv_3 -1$, and ***3Pos*** if $k \equiv_3 +1$. Note $85 = 5 \cdot 17$ is a product of two 3Neg primes. We simply need to place one prime in J and the other in K. Hence a solution is $(J, K) := (5 \cdot 5,\ 17 \cdot 17)$.

A more general soln is $(J, K) := (5p,\ 17q)$. where p, q are 3Neg numbers st. $p \nmid 17$ and $q \nmid 5$. Letting $p = q := 2$ yields $(J, K) := (10,\ 34)$ as the smallest soln.

Also, $\mathcal{T}$-GCD$(175, 70) =$ 7. $175 = 7{\cdot}5{\cdot}5$ and $70 = 7{\cdot}5{\cdot}2$.

10 10 **c** On a K-elt set Ω, the number $\#_K$ of **reflexive symmetric** binrels is $2^{\binom{K}{2}} = 2^{\frac{[K-1]K}{2}}$.

In particular, $\#_5 =$ ______.

Counting: A refl-binrel owns all pairs (μ, μ), for $\mu \in \Omega$. The # of 2-sets $\{\alpha, \beta\}$ is $\binom{K}{2}$. For each 2-set, either both pairs (α, β) and (β, α) are <u>*in*</u> the *symmetric* relation, or both are <u>*out*</u>; **two** choices, whence $\mathbf{2}^{\binom{K}{2}}$ refl-symm binrels.

Hence: $\#_5 = 2^{\binom{5}{2}} = 2^{10} = 1024$.

15 **d** On a 3-set, there are 5 many equiv.relations.

Partitions. Let P_n be the number of ptns having precisely n nv-atoms. Then $P_1 = \binom{3}{3} = 1$, $P_2 = \binom{3}{1,2} = 3$, $P_3 = \binom{3}{1,1,1}/3! = 1$.

URL https://en.wikipedia.org/wiki/Partition_of_a_set has examples of Partition Pictures. ♦

10 10 e Multinomial coefficient $\binom{9}{4,2,3} = \underline{\hspace{4cm}} = \underline{\hspace{2cm}}$.

[Write your answer as a product of binomial coeffs, then compute the product as a single integer,]

Nomial Soln: Directly, $\binom{9}{4,2,3} = \frac{9!}{4!\cdot 2!\cdot 3!}$. Computing, $\binom{9}{4,2,3} = \binom{9}{4}\cdot\binom{5}{2}\cdot\binom{3}{3} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{5\cdot 4}{2\cdot 1}$.

Hence, $\binom{9}{4,2,3} = 9{\cdot}2{\cdot}7\cdot 5{\cdot}2 = [63\cdot 2]\cdot 10 = 1260$.

Lisp code:

```
(multinom-coeff 9 '(4 2 3))  =>  1260
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OYOP: *In grammatical English* ***sentences***, *write each essay on every 2nd line* (usually), *so that I can easily write between the lines.*

B5: Consider a strict well-order $\prec$ on set $\mathbf{U}$, and a strict well-order $<$ on $\boldsymbol{\Gamma}$. Define binrel $\ll$ on $\mathbf{U}\times\boldsymbol{\Gamma}$ by:

$$(b,\alpha) \ll (c,\beta)$$

IFF *Either* $b \prec c$ *or* $[b = c$ and $\alpha < \beta]$.

Prove: *Relation* $\ll$ *is a well-order on* $\mathbf{U}\times\boldsymbol{\Gamma}$.
[*You may assume that* $\ll$ *is a total-order.*]

Well-order: Consider a non-empty subset $Q \subset \mathbf{U}\times\boldsymbol{\Gamma}$. Extracting the 1st-elt of each ordered-pair, the set

$$\left\{ b \in \mathbf{U} \;\middle|\; \exists\alpha \in \boldsymbol{\Gamma} \text{ with } (b,\alpha) \in Q \right\}$$

is non-void, hence has a $\prec$-minimum elt; call it $\boldsymbol{m}$.

Extracting 2nd-elements, the set

$$\left\{ \beta \in \boldsymbol{\Gamma} \;\middle|\; (\boldsymbol{m},\beta) \in Q \right\}$$

is non-void, so *it* has a $<$-minimum elt which we'll call $\boldsymbol{\mu}$.

THE UPSHOT: Pair $(\boldsymbol{m},\boldsymbol{\mu})$ is the $\ll$-minimum element of subset Q.

B6: Define: "On a set E, a binary relation ∇ is an ***equivalence relation*** IFF...". *Make sure* to define any terms like "reflexive" that you use in your defn.!

Let $\mathbb{P}$ be the set of ordered integer-pairs (n, d), with $d \neq 0$. Define relation $\boldsymbol{C}$ on $\mathbb{P}$ by

$$(N, D) \ \boldsymbol{C} \ (x, y) \qquad \text{IFF} \qquad N \cdot y = x \cdot D.$$

Prove, in detail, that $\boldsymbol{C}$ is an equivalence relation.
[Only work in $\mathbb{Z}$; do ***not*** use fractions.]

B4:	___ ___	95pts
B5:	___ ___	45pts
B6:	___ ___	45pts
Total:	___ ___ ___	185pts

NAME: ....

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: *Energetic Student* ....