Complex Analysis Problem List α Prof. J. King

Notation. ***ITOf***, in terms of; ***RoU***, Root(s) of Unity; ***Poly(s)***, polynomial(s); ***fnc(s)***, function(s). Let $f: B \hookleftarrow$ mean $f: B \to B$. Use $\Im(z)$ and $\Re(z)$ for the imaginary and real parts of z.

Let $\mathbb{S}$ be the Riemann Sphere, the set of (a, b, c) with $a^2 + b^2 + c^2 = 1^2$; use $\mathbb{NP} := (0, 0, 1)$ and $\mathbb{SP} := (0, 0, -1)$ for the North/South Poles. As usual, positive real axis is identified with the pos. a-axis, and the imaginary axis with the b-axis. Let $\sigma: \mathbb{C}^* \to \mathbb{S}$ be stereographic projection. We showed that

(1) $\sigma(z) = \frac{1}{M}\left(z + \bar{z}, \frac{1}{\imath}[z - \bar{z}], z\bar{z} - 1\right)$, where $M = z\bar{z} + 1$,

and $\sigma(\infty) = \mathbb{NP}$.

Suppose that functions $f: \mathbb{S} \hookleftarrow$ and $F: \mathbb{C}^* \hookleftarrow$ are related by stereographic projection; that is, $f = \sigma \circ F \circ \sigma^{-1}$. We say that each is the ***corresponding function*** of the other.

Let $\mathbb{U}$ denote both the unit circle in $\mathbb{C}^*$ and the equator in $\mathbb{S}$. When needed, we think of $\mathbb{U}$ as positively (anticlockwise) oriented.

Some of the assigned HW problems may appear on the exam; ditto E1 or E3-E7. Make sure that you can do probs 1–17 on P. 79 of our text.

Geometry

1: Prove the Cauchy-Schwarz Inequality, $\Re(z\bar{w}) \le |z||w|$ with equality IFF *what*? Now use C-S to prove the Triangle Inequality.

2: Let θ be the angle from z to w (each a non-zero complex number), and let $u := \operatorname{cis}(\theta)$. Express u as in terms of z and w.

3: Show that $p, q, r \in \mathbb{C}$ are vertices of an equilateral triangle IFF $p^2 + q^2 + r^2 = pq + qr + pr$.

4: For each triangle $p, q, r \in \mathbb{C}$, express the center and radius (C & R) of the circumscribed circle. Write your expressions symmetrically in p, q, r.

5: For $S \in \mathbb{R}$ and $J \in \mathbb{C}_{\neq 0}$, show that eqn.

$$\Re(z\bar{J}) = S$$

describes a line. Draw a picture and interpret S and J geometrically. [H: First consider $J \in \mathbb{U}$.]

Now suppose that $\Re(z\bar{K}) = T$ is another line, not parallel to the first. (How can you express non-parallelness in terms of J, K, S, T?) Express the intersection P of these two lines ITOf J, K, S, T.

Stereographic projection

6: Here, a **"π-rotation"** of the Riemann Sphere means rotating $\mathbb{S}$, by π radians, about some line through the origin. (i) Let f be π-rotation about the a-axis. [So $f((a, b, c)) = (a, -b, -c)$.] Compute the corresponding map $F: \mathbb{C}^* \hookleftarrow$.

(ii) For a point $u = (a, b, 0)$ on the equator, let f_u π-rotate the sphere about the line through u. So $f_{(1,0,0)}$ equals the f above. Compute the corr. map F_u. (iii) Use the foregoing to show that a composition of any number of π-rotations about lines through the equator, equals a rotation (by some angle) about some line. Can you generalize so as to remove the words "lines through the equator"?

7: Prove that stereographic projection engenders a 1-to-1 correspondence between circles on $\mathbb{S}$ and generalized circles on $\mathbb{C}^*$.

8: On $\mathbb{S}$, consider the circle $a + b + c = \frac{1}{3}$. Compute the center and radius of the corresponding "circle" on $\mathbb{C}^*$. (b) As a function of $D \in \left[-\sqrt{3}, \sqrt{3}\right]$, let $r(D)$ be the radius of the $\mathbb{C}^*$-circle corresponding to the $\mathbb{S}$-circle with $a + b + c = D$. Compute $r(D)$.

Polynomials and roots

Let $\mathrm{RoU}(N)$ be the set of Nth roots of unity, and let $\mathrm{Prim}(N)$ be the subset of primitive Nth roots.

9: For $N = 1, 2, \ldots$, compute the sum and product of the Nth RoU.

10: Let $\mathrm{Prod}(N)$ be the product of the *primitive* Nth RoU. Show that $\mathrm{Prod}(\mathrm{odd}) = 1$. How does $\mathrm{Prod}(\mathrm{even})$ behave? [H: A picture makes this easy.]

11: For $N = 1, 2, \ldots, 12$, compute *sum* of the *primitive* Nth RoU.

12: Let $P(z) \coloneqq z^4 + 3z^2 + 5$ and $D(z) \coloneqq z^2 + z + 3$. Compute polys Q and R such that $P = D \cdot Q + R$, with $\deg(R) < \deg(D)$.

13: (a) Suppose that 17 is a multiplicity-3 root of poly $P(z)$. Show that $z - 17$ is a factor of $P'''(z)$, but not of $P^{(4)}(z)$. Generalize.

(b) For a poly P with kth derivative $P^{(k)}$, what general statement can you make about the greatest common divisor poly $G \coloneqq \gcd\{P, P^{(k)}\}$?

14: Compute the polynomial

$$P_N(z) \coloneqq \prod_{r \in \mathrm{RoU}(N)} [z - r],$$

for $N = 1, 2, \ldots, 12$. Now do the same for

$$\Psi_N(z) \coloneqq \prod_{r \in \mathrm{Prim}(N)} [z - r].$$

[H: Efficiently, this can be done recursively. Can you argue, for instance, that the product $\Psi_{10}\Psi_5\Psi_2\Psi_1$ equals P_{10}?]

15: Argue that Ψ_N is always a monic polynomial with *integer* coefficients. Is it now easy to argue that the sum of the primitive Nth roots is always an integer?

Limits and Differentiation

Abbreviate a sequence $(w_n)_{n=1}^{\infty}$ by $\vec{w}$, and let $\lim(\vec{w})$ mean $\lim_{n\to\infty} w_n$. For a function f, let $f(\vec{w})$ abbreviate the sequence $\big(f(w_n)\big)_{n=1}^{\infty}$.

An increasing sequence $n_1 < n_2 < \ldots$ of indicies determines a subsequence $\vec{u}$ with $u_k \coloneqq w_{n_k}$. We write $\vec{u} \subset \vec{w}$ to indicate that u is a subsequence of w.

16: Compute $u(z) \coloneqq \Re(\exp(z^2))$. Directly show that u is harmonic.

17: [Below, use $c()$ and $s()$ to abbr. cos() and sin().] Which of the below fncs u is harmonic? For those which are, compute the conj-harmonic fnc v. Now identify the analytic fnc $f \coloneqq u + iv$.

(a) $u \coloneqq x^4 - x^2[1 + 6y^2] + y^2[1 + y^2]$.

(b) $u \coloneqq e^{2x} c(2y)$.

(c) $u \coloneqq x/[x^2 + y^2]$.

18: Suppose h is analytic. State the Cartesian and Polar forms of the C-R eqns., using h_x, h_y, h_r and h_θ.

19: Show that $\lim(\vec{w}) = 5$ if: *Each subseq.* $\vec{v} \subset \vec{w}$ *has a further subseq.* $\vec{u} \subset \vec{v}$ *such that* $\lim(\vec{u}) = 5$.

(b) For a fnc f on $\mathbb{C}_{\neq 0}$, show that $\lim_{z\to 0} f(z) = 5$ if the following holds for each sequence $\vec{z} \subset \mathbb{C}_{\neq 0}$ which converges to 0: *Each subsequence of* $\vec{z}$ *has a further subseq.* $\vec{u}$ *so that* $\lim\big(f(\vec{u})\big)$ *equals* 5.

(c) Prove the Chain Rule: *Suppose g and h are functions such that h is differentiable at 0 and g is differentiable at $P \coloneqq h(0)$. Then $f \coloneqq g \circ h$ is differentiable at 0, and*

$$(*) \qquad f'(0) = g'(P) \cdot h'(0).$$

[H: Say that a number $u \in \mathbb{C}_{\neq 0}$ is bad if $h(u) = h(0)$. Now, Given an arbitrary subseq. $\vec{v}$ of a sequence converging to 0, argue that there is a further subseq. $\vec{u} \subset \vec{v}$ which is always bad or never bad. In either case, argue now that it is easy to establish $(*)$ using (b).]

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