Complex Analysis Exam Alpha Prof. J. King

Notation. Let $\mathbb{S}$ be the Riemann Sphere, the set of (a,b,c) with $a^2+b^2+c^2=1^2$; use $\mathbb{NP} := (0,0,1)$ and $\mathbb{SP} := (0,0,-1)$ for the North/South Poles. As usual, the positive real axis is identified with the positive a-axis, and the imaginary axis with the b-axis. Let $\sigma\colon \mathbb{C}^* \to \mathbb{S}$ be stereographic projection. We showed that

(1) $\sigma(z) = \frac{1}{M}\big(z+\bar{z}, \frac{1}{\imath}[z-\bar{z}], z\bar{z}-1\big)$, where $M = z\bar{z}+1$,

and $\sigma(\infty) = \mathbb{NP}$.

α1: Quickies [30 points per question]. Fill in the blanks below, writing your answer in the simplest form unless otherwise indicated. **Show no work.** There is no partial credit for this question, so carefully check that you have written what you mean. Make sure that you write expressions unambiguously e.g, the expression "$1/a+b$" should be parenthesized either $(1/a)+b$ or $1/(a+b)$ so that I know your meaning. Similarly, write "$\sin(x)$" rather than "$\sin x$".

(a) Compute: $\operatorname{Log}(2\imath)=$ ______ .

(b) Give a $z = x+\imath y$ such that $\operatorname{Log}\big(\exp(z)\big) \neq z$.
$x=$ ______ & $y=$ ______ .

(c) Let w be a value of $\log(\imath^2)$ which is *not* a value of $2\log(\imath)$. $w=$ ______ .

(d) Give a fnc $f\colon \mathbb{C} \hookleftarrow$ with continuous first partials, yet f is nowhere analytic and nowhere anti-analytic.
$f(z)=$ ______ .

(e) Write K, the number of primitive million-th roots of unity, as a product of powers of distinct primes.
$K=$ ______ .

(f) $\Re\Big(e^{e^{x+\imath y}}\Big)=$ ______ .

(g) Compute the real number $\alpha=$ ______ such that

$$3^{\alpha}\sum_{k=0}^{4000}\binom{4000}{k}2^k = \sum_{j=0}^{1995}\binom{1995}{j}8^j\,.$$

α2: Short Answer [50 pts per question]. Fill in the blanks below. **Show no work.**

(h) Let $u(x,y)$ be $x^3-3xy^2-x^2+y^2$. Compute the conjugate harmonic function
$v(x,y)=$ ______
and identify the analytic fnc $f := u+\imath v$.
$f(z)=$ ______ .

(i) On $\mathbb{S}$, consider the circle $a+b+c=\frac{1}{3}$. Compute the center and radius of the corresponding "circle" on $\mathbb{C}^*$. Center= ______ & Radius= ______ .

As a function of $D \in \big[-\sqrt{3}, \sqrt{3}\big]$, let $S(D)$ be the *square*-of-the-radius of the $\mathbb{C}^*$-circle corresponding to the $\mathbb{S}$-circle with $a+b+c=D$.
Compute $S(D)=$ ______ .

For the next two questions, show all work on separate sheets of paper. Draw good large pictures to help explain your arguments.

α3: [90 pts] For a fnc $h\colon \mathbb{C} \hookleftarrow$, state the Cartesian and the Polar forms of the Cauchy-Riemann eqns., using h_x, h_y, h_r and h_θ.

Now define a function $f\colon \mathbb{C} \hookleftarrow$ by

$$f(z) := \begin{cases} (\bar{z})^2/z & \text{if } z \text{ not zero;} \\ 0 & \text{if } z = 0. \end{cases}$$

Explicitly compute f_x and f_y at the origin. At the origin, show that f fulfills the Cartesian C-R eqns.. Prove or disprove: *This f is differentiable at the origin.*

α4: [40 pts] Precisely state an interesting complex analysis problem that you expected to be asked, but weren't. Now carefully solve this problem, using pictures (if possible) as part of your explanation.

I will grade your problem both on its quality and on the correctness and elegance of your solution. The question may be difficult, but avoid difficulties that are uninteresting. E.g, *"Compute the real part of*

$$\sin\Big(\sin\big(\ldots \text{10 times} \ldots (\sin(z))\big)\Big)\text{"}$$

is difficult, without being interesting.