Two-page Technique (Map / Computation)

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Challenge: Compute Re and Im of $\sum_{k=3}^{88} [1+\mathbf{i}]^k$.

$\mathcal{M}$: Initial plan:

a: Give symbolic names to quantities. If reasonable, draw the Argand plane with $[1+\mathbf{i}]$ plotted; maybe also, the first few $[1+\mathbf{i}]$ powers plotted.

b: Remember, or derive, how to sum geometric series.

Letting $\mathsf{B} := [1+\mathbf{i}]$, $U := 88$ and $L := 3$, we seek to compute

†: $$\mathsf{S} := \sum_{k=L}^{U} \mathsf{B}^k .$$

We learned geometric series in form $\sum_{n=0}^{N} R^n$. So we rewrite (†) as

‡: $$\mathsf{S} = \sum_{k=L}^{L+N} \mathsf{B}^k \stackrel{\text{note}}{=\!=} \mathsf{B}^L \cdot \sum_{n=0}^{N} \mathsf{B}^n ,$$

where $N := U - L$.

$\mathcal{C}$: With $N = U - L = 88 - 3 = 85$, we will compute

‡c: $$\mathsf{S} = [1+\mathbf{i}]^3 \cdot \sum_{n=0}^{85} [1+\mathbf{i}]^n .$$

$\mathcal{M}$: Alas, I don't remember the formula for $\sum_{n=0}^{N} R^n$, so I'll derive it. First, let's name this sum; can't use S [already in use], so I'll use T. With $T := \sum_{n=0}^{N} R^n$, note

$$\begin{aligned}[R-1]\cdot T &= RT - T \\ &= R^{N+1} + R^N + R^{N-1} + \ldots + R^2 + R \\ &\quad - [R^N + R^{N-1} + \ldots + R^2 + R + 1] \\ &\stackrel{\text{note}}{=\!=} R^{N+1} - 1 .\end{aligned}$$

If $R \neq 1$, we may divide by $R - 1$, giving

∗: $$T = \frac{R^{N+1} - 1}{R - 1}.$$

$\mathcal{C}$: Since $[1+\mathbf{i}] \neq 1$, we may use (∗), giving

∗c: $$\begin{aligned}\mathsf{S} &= [1+\mathbf{i}]^3 \cdot \frac{[1+\mathbf{i}]^{85+1} - 1}{[1+\mathbf{i}] - 1} \\ &\stackrel{\text{note}}{=\!=} [1+\mathbf{i}]^3 \cdot \frac{[1+\mathbf{i}]^{86} - 1}{\mathbf{i}} \\ &\stackrel{\text{note}}{=\!=} -\mathbf{i} \cdot [1+\mathbf{i}]^3 \cdot \Big[[1+\mathbf{i}]^{86} - 1\Big] .\end{aligned}$$

$\mathcal{M}$: We've reduced the Challenge to computing a power of a complex-number. Writing the number in polar form as $re^{\mathbf{i}\theta}$ with $r \geq 0$ and $\theta \in \mathbb{R}$, its K^{th}-power is

∗∗: $$\left[re^{\mathbf{i}\theta}\right]^K = r^K \cdot \exp(\mathbf{i}\, K\theta) .$$

$\mathcal{C}$: We *could* use (∗∗) to finish. Here, the specific base B allows us to proceed differently. We notice that $[1+\mathbf{i}]^2 = 2\mathbf{i}$. Thus $\mathsf{B}^{86} = 2^{43} \cdot \mathbf{i}^{43}$. Multipl-by-$\mathbf{i}$ is periodic, with period 4, so $\mathbf{i}^{43} = -\mathbf{i}$. Thus,

$$[1+\mathbf{i}]^{86} - 1 = -[1 + 2^{43}\mathbf{i}] .$$

Our (∗c) hands us

$$\mathsf{S} = \mathbf{i} \cdot [1+\mathbf{i}]^3 \cdot [1 + \mu\mathbf{i}] , \quad \textit{where}$$

we've abbreviated the multiplier by $\mu := 2^{43}$.

LAST STEP: A bit of elbow-grease yields

$$\begin{aligned}[1+\mathbf{i}]^3 &= \mathbf{i}^3 + 3\mathbf{i} + 3\mathbf{i}^2 + 1 = 2\mathbf{i} - 2 . \quad \textit{So} \\ \mathbf{i} \cdot [1+\mathbf{i}]^3 &= 2 \cdot [-1 - \mathbf{i}] .\end{aligned}$$

Consequently,

$$\mathsf{S} = 2 \cdot [-1 - \mathbf{i}] \cdot [1 + \mu\mathbf{i}] .$$

Energetic Reader can now multiply this out, compute real and imaginary parts, and –*lastly*– substitute 2^{43} for μ as the final step. *Nifty cool!*

C: Easily, $[-1 - \mathbf{i}] \cdot [1 + \mu\mathbf{i}] = [\mu - 1] - [\mu + 1]\mathbf{i}$. Our μ is *real*, and thus

$$\operatorname{Re}(\mathsf{S}) = 2\cdot[\mu - 1] = 2^{44} - 2, \quad \textit{and}$$

$$\operatorname{Im}(\mathsf{S}) = 2\cdot[-\mu - 1] = -2^{44} - 2.$$

An Elegant answer to a Nice problem ...