Complex Analysis homework.

by *Energetic Plex Student*, Aug. 2021

[See source file for additional info]

Most of $\mathbb{C}$. Use $\mathbf{P} := \text{PBal}_\infty(0)$ for the Punctured plane, and $\mathbf{L} := \mathbb{C} \smallsetminus (-\infty, 0]$ for the sLit-plane. □

1: Log-Abs Lemma. *Functions $\mathsf{u}(z) := \ln(|z|)$ on $\mathbf{P}$, and $\mathsf{v}(z) := \text{Arg}(z)$ on $\mathbf{L}$, are harmonic.* ♢

Pf abstract. On $\mathbf{P}$, fnc Log has real and imaginary parts $\mathsf{u}(z) := \ln(|z|)$ and $\mathsf{v}(z) := \text{Arg}(z)$. Locally, Log is holomorphic except for the jump-discty of Arg on the negative real-axis. Consequently, u is harmonic on $\mathbf{P}$, and v is harmonic on the slit-plane. ♦

Pf direct. Let $S := x^2 + y^2$. Then $\mathsf{u} = \ln(S^{1/2})$, giving

$$\mathsf{u}_x = \frac{1}{S^{1/2}} \cdot \frac{1/2}{S^{1/2}} \cdot 2x \overset{\text{note}}{=\!=} \frac{x}{S}. \quad \textit{Thus,}$$

$$\mathsf{u}_{xx} = \frac{1 \cdot S - x \cdot 2x}{S^2} = \frac{y^2 - x^2}{S^2}.$$

Symmetry gives $\mathsf{u}_{yy} = \frac{x^2-y^2}{S^2}$, so $\mathsf{u}_{xx} + \mathsf{u}_{yy} = 0$. ♦

Pf direct. [*Argument is slightly annoying due to the arbitrary choices,* arctan() on $(-\frac{\pi}{2}, \frac{\pi}{2})$ *and* Arg() on $(-\pi, \pi]$, *in the relevant defns.*]

For $x>0$, note $\mathsf{v}(z) = \arctan(y/x)$. The Chain rule asserts

$$\mathsf{v}_y = \frac{1}{1+[y/x]^2} \cdot \frac{1}{x} \overset{\text{note}}{=\!=} \frac{x}{S} \quad \textit{and}$$

$$\mathsf{v}_x = \frac{1}{1+[y/x]^2} \cdot \frac{-y}{x^2} \overset{\text{note}}{=\!=} -\frac{y}{S}. \quad \textit{Diff'ing again,}$$

$$\mathsf{v}_{xx} = [-y] \cdot \frac{-1}{S^2} \cdot 2x \overset{\text{note}}{=\!=} \frac{2xy}{S^2}.$$

After negation, v_y is symmetric with v_x. Consequently, $\mathsf{v}_{yy} = -\frac{2xy}{S^2}$, showing v to be harmonic.

[Cases $x<0$ and $x=0$ are left to the reader.] ♦

2: H-H Lemma. *Suppose $\mathbb{R} \overset{h}{\leftarrow} \Lambda \overset{f}{\leftarrow} \Omega$, where Λ,Ω are open subsets of $\mathbb{C}$. If h is harmonic and f holomorphic, then $h \circ f$ is harmonic.* ♢

Pf abstract. Fix $p \in \Omega$ and an open ball $\mathsf{B} \subset \Omega$ about $f(p)$. As f is cts, its inverse-image $f^{-1}(\mathsf{B})$ is open. So:

It suffices to show $\boxed{h \circ f \text{ is harmonic on } f^{-1}(\mathsf{B})}$.

On B, our h has an harmonic conjugate $\widetilde{h}$, since B *is simply-connected*. On B, then, $H := h + \mathbf{i}\widetilde{h}$ holomorphic, whence $H \circ f$ is holomorphic on $f^{-1}(\mathsf{B})$.

Thus $\text{Re} \circ H \circ f \overset{\text{note}}{=\!=} h \circ f$ is harmonic on $f^{-1}(\mathsf{B})$. ♦

Pf computation. Let $f := \mathsf{u} + \mathbf{i}\mathsf{v}$ and $G := h \circ f$, i.e,

$$G(x,y) := h\Big(\mathsf{u}(x,y), \mathsf{v}(x,y)\Big).$$

Then

$$G_x = h_\mathsf{u} \cdot \mathsf{u}_x + h_\mathsf{v} \cdot \mathsf{v}_x. \quad \textit{Thus}$$

$$G_{xx} = [h_{\mathsf{uu}}\mathsf{u}_x + h_{\mathsf{uv}}\mathsf{v}_x] \cdot \mathsf{u}_x + h_\mathsf{u} \cdot \mathsf{u}_{xx}$$
$$+ [h_{\mathsf{vu}}\mathsf{u}_x + h_{\mathsf{vv}}\mathsf{v}_x] \cdot \mathsf{v}_x + h_\mathsf{v} \cdot \mathsf{v}_{xx}. \quad \textit{Similarly,}$$

$$G_{yy} = [h_{\mathsf{uu}}\mathsf{u}_y + h_{\mathsf{uv}}\mathsf{v}_y] \cdot \mathsf{u}_y + h_\mathsf{u} \cdot \mathsf{u}_{yy}$$
$$+ [h_{\mathsf{vu}}\mathsf{u}_y + h_{\mathsf{vv}}\mathsf{v}_y] \cdot \mathsf{v}_y + h_\mathsf{v} \cdot \mathsf{v}_{yy}.$$

Sum $h_\mathsf{u} \cdot \mathsf{u}_{xx} + h_\mathsf{u} \cdot \mathsf{u}_{yy} = h_\mathsf{u} \cdot [\mathsf{u}_{yy} + \mathsf{u}_{xx}] = 0$, since u is harmonic. And v is harmonic, so $G_{xx} + G_{yy}$ equals

$$h_{\mathsf{uu}}\mathsf{u}_x\mathsf{u}_x + h_{\mathsf{uv}}\mathsf{v}_x\mathsf{u}_x + h_{\mathsf{vu}}\mathsf{u}_x\mathsf{v}_x + h_{\mathsf{vv}}\mathsf{v}_x\mathsf{v}_x$$
$$+ h_{\mathsf{uu}}\mathsf{u}_y\mathsf{u}_y + h_{\mathsf{uv}}\mathsf{v}_y\mathsf{u}_y + h_{\mathsf{vu}}\mathsf{u}_y\mathsf{v}_y + h_{\mathsf{vv}}\mathsf{v}_y\mathsf{v}_y.$$

Our C-R eqns permit replacing each v_y by u_x, and each v_x by $[-\mathsf{u}_y]$. The rewritten ΔG is

$$h_{\mathsf{uu}}\mathsf{u}_x\mathsf{u}_x + h_{\mathsf{uv}}[-\mathsf{u}_y]\mathsf{u}_x + h_{\mathsf{vu}}\mathsf{u}_x[-\mathsf{u}_y] + h_{\mathsf{vv}}[-\mathsf{u}_y][-\mathsf{u}_y]$$
$$+ h_{\mathsf{uu}}\mathsf{u}_y\mathsf{u}_y + h_{\mathsf{uv}}\mathsf{u}_x\mathsf{u}_y + h_{\mathsf{vu}}\mathsf{u}_y\mathsf{u}_x + h_{\mathsf{vv}}\mathsf{u}_x\mathsf{u}_x.$$

The rightmost blue term above green cancel out. The leftmost cyan above red cancel. So ΔG equals

$$h_{\mathsf{uu}}\mathsf{u}_x\mathsf{u}_x + h_{\mathsf{vv}}\mathsf{u}_y\mathsf{u}_y + h_{\mathsf{uu}}\mathsf{u}_y\mathsf{u}_y + h_{\mathsf{vv}}\mathsf{u}_x\mathsf{u}_x$$
$$= [h_{\mathsf{uu}} + h_{\mathsf{vv}}] \cdot \Big[[\mathsf{u}_x]^2 + [\mathsf{v}_x]^2\Big].$$

The leftmost bracket is zero, since h is harmonic. ♦