Complex Analysis homework.

by *Energetic Plex Student*, April 2023

jk#Laurent series[P]**.** Compute the power-series $\sum_{n=0}^{\infty} B_n z^n$ for $F(z) := \frac{1}{z-3}$, on ball $\mathcal{B} := \text{Bal}_3(0)$.

For $F()$, compute Laurent series $\sum_{n\in\mathbb{Z}} A_n z^n$ on annulus $\mathcal{A} := \text{Ann}_{\infty}^{3}(0)$.

Ball. Using a convergent geometric series: For $z \in \mathcal{B}$,

$$F(z) = \frac{1}{-3}\cdot\frac{1}{1-\frac{z}{3}} = \frac{1}{-3}\cdot\sum_{n=0}^{\infty}\left[\frac{z}{3}\right]^n. \quad \textit{Hence,}$$

$$\dagger: \quad B_{n\geq 0} = \frac{-1}{3^{n+1}} \quad \textit{and} \quad B_{n<0} = 0.$$ ♦

Annulus. For $z \in \mathcal{A}$,

$$F(z) = \frac{1}{z}\cdot\frac{1}{1-\frac{3}{z}} = \frac{1}{z}\cdot\sum_{k=0}^{\infty}\left[\frac{3}{z}\right]^k. \quad \textit{Hence,}$$

$$\ddagger: \quad A_{n\geq 0} = 0 \quad \textit{and} \quad A_{n<0} = \frac{1}{3^{n+1}}.$$ ♦

Coeff formula. Centered at $\boldsymbol{z_0}$, thm 8.25, P.120 of FiCA says

$$g(z) = \sum_{n\in\mathbb{Z}} \text{Coef}_n \cdot [z - \boldsymbol{z_0}]^n, \quad \textit{where}$$

$$\text{Coef}_n = \frac{1}{2\boldsymbol{\pi}\mathbf{i}}\oint_{\mathsf{C}} \frac{g(w)}{[w-\boldsymbol{z_0}]^{n+1}}\,\text{d}w,$$

for appropriate contour C, with z in the correct set.□

Ball via residues. In our case, the above integrand is $H_n(w) := \frac{1}{w-3}\Big/w^{n+1}$. Thus

$$\text{Coef}_n = \begin{bmatrix}\text{Sum of } H_n \text{ residues}\\ \text{enclosed by } \mathsf{C}.\end{bmatrix}$$

Looking ahead, $F^{(n)}(z) = \frac{\prod_{k=1}^{n}[-k]}{[z-3]^{n+1}} = \frac{n!\cdot[-1]^n}{[z-3]^{n+1}}$. Computing

$$*: \quad \begin{aligned} &\textit{For } n\geq 0\text{:} \quad \text{Res}(H_n,0) \overset{\text{CIF}}{=\!=} \frac{F^{(n)}(0)}{n!} = \frac{-1}{3^{n+1}}. \\ &\textit{For } n<0\text{:} \quad \text{Res}(H_n,0) = 0, \quad \text{since } H_n \text{ is holomorphic at } 0. \end{aligned}$$

This agrees with the B_n coefficients from (†), above.♦

Annulus via residues. Now our SCC C encloses *both* 3 and 0, so we sum residues at both. By CIF,

$$\textit{For } n\in\mathbb{Z}\text{:} \quad \text{Res}(H_n,3) = \frac{1}{3^{n+1}}.$$

Now $A_n = \text{Res}(H_n,3) + \text{Res}(H_n,0)$, so (∗) gives

$$A_n = \frac{1}{3^{n+1}} + \begin{Bmatrix}\frac{-1}{3^{n+1}} & \textit{, when } n\geq 0\\ 0 & \textit{, when } n<0\end{Bmatrix} = \begin{Bmatrix}0 & \textit{, when } n\geq 0\\ \frac{1}{3^{n+1}} & \textit{, when } n<0\end{Bmatrix}.$$

And this agrees with (‡). ♦