Staple!

Sets and Logic
MHF3202 7860

Individual Optional Project-Z

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31Aug2015

This IOP is due **noon, Thurs, 23Apr2015**, slid *completely* under my office door, 402 LITTLE HALL.

This sheet is "Page $1/N$", and you've labeled the rest as "Page $2/N$",..., "Page N/N". Fill-in [large handwriting] *on this problem-sheet* all of your blanks.

Please write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

Your 3 essay(s) must be TYPESET, *and Double or Triple spaced. Use the Print/Revise cycle to produce good, well thought out, essays. Start each essay on a new sheet of paper. Do **not** restate the problem; just solve it.*

Nomenclature. Below, a ***blip*** is an *infinite* subset of $\mathbb{Z}$. Use $\mathcal{P}_\infty$ for the collection of *all* blips. For a collection, $\mathcal{C}$, of blips, each element of $\mathcal{C}$ is a "$\mathcal{C}$-***blip***".

Z1: A blip-collection, $\mathcal{C}$, is 5-***bounded*** if:

For all distinct $\mathcal{C}$-blips X and Y, necessarily $|X \cap Y| \leqslant 5$.

Prove that each 5-bounded blip-collection is only countable. [*Hint:* Find a known countable-set $\mathbf{\Omega}$, and construct an *injection* $f: \mathcal{C} \hookrightarrow \mathbf{\Omega}$. Note: Each of the following sets is countable: $\mathbb{N}\times\mathbb{N}$. *A countable union of countable-sets.* $\mathbb{Q}$. *The collection of finite subsets of a countable-set. The set of intpolys.*]

Z2: A blip-collection, $\mathcal{B}$, is ***intersection-finite*** if:

For all distinct $X, Y \in \mathcal{B}$, *necessarily* $|X \cap Y| < \infty$.

Construct, with proof, a *particular* intersection-finite blip-collection $\mathcal{B}$, with continuum-cardinality. I.e, $\mathcal{B} \asymp \mathbb{R}$.
[*Hint:* Find a known continuum-cardinality set $\mathbf{U}$. Create an *injection* $g: \mathbf{U} \hookrightarrow \mathcal{P}_\infty$ st.: $\forall x, y \in \mathbf{U}$, if $x \neq y$ then $\left|g(x) \cap g(y)\right| < \infty$. Then $\mathcal{B} := \text{Range}(g)$ is a soln. NB: The following sets have continuum-cardinality: *Each non-degenerate interval.* $\mathbb{R}\times\mathbb{R}$. *The set of transcendental numbers.* $\mathcal{P}_\infty$. *The set of real-coeff polynomials.*]

Z3: Recall:

Defn. For a prime p and integer z, the ***Legendre-symbol*** is written as

$$\left(\frac{z}{p}\right) \quad \text{or, in email, also as} \quad (z \,/\!/\, p)\,.$$

By defn, $\left(\frac{z}{p}\right)$ is +1, if $z \in \mathrm{QR}_p$; is −1, if $z \in \mathrm{NQR}_p$; and is 0, if $z \not\perp p$, i.e $z \bullet p$.

An odd integer k is "**4Pos**" if $k \equiv_4 +1$; is 4**Neg** if $k \equiv_4 -1$; is 8**Near** if $k \equiv_8 \pm 1$ (either); is 8**Far** if $k \equiv_8 \pm 3$. □

1: Legendre-symbol Thm. *Fix an odd prime p and $H := \frac{p-1}{2}$. Use $\langle\cdot\rangle_p$ for symmetric residue, selecting from $[-H \,..\, H]$. For each integer z:*

a: *The (symmetric) residue $\langle z^H \rangle_p$ equals* $\left(\frac{z}{p}\right)$.

b: *For x and z integers:* $\left(\frac{x}{p}\right) \cdot \left(\frac{z}{p}\right) = \left(\frac{xz}{p}\right)$. *I.e, the mapping $x \mapsto \left(\frac{x}{p}\right)$ is totally-multiplicative.*

c: *Value −1 is a QR_p IFF p is 4Pos. And (courtesy Wilson's Thm), value $r := [H!]$ is a mod-p sqroot of −1.*

d: *The number 2 is a p-QR IFF p is 8NEAR, that is, $p \equiv_8 \pm 1$.* ◊

Your goal is to prove:

†: The Sixteen Thm *For each oddprime p, the congruence $x^8 \equiv_p 16$ admits a solution.*

In your WU, you may use $\sim$ for $\equiv_4$ and $\approx$ for $\equiv_8$, if you wish. But use $\equiv_p$ or $\equiv$ for congr-mod-p.

i FTSOC, suppose you have a p with *no* solution to $x^8 \equiv_p 16$. Prove that $2 \in \mathrm{NQR}_p$ and $-1 \in \mathrm{QR}_p$. Use LSThm to compute $\langle p \rangle_8$ as a non-negative residue.

ii Let r be a p-sqroot of −1. Use LST to prove that $r \in \mathrm{QR}_p$. But use a different part of LST to prove that $r \in \mathrm{NQR}_p$. *Contradiction*, QED.

iii SHORT ANSWER: Give an example of a 2 digit prime $q :=$ ____ with $2 \in \mathrm{NQR}_q$ and $-1 \in \mathrm{QR}_q$. Using symmetric residues, $\mathrm{QR}_q = \{$ ____ $\}$ and $\mathrm{NQR}_q = \{$ ____ $\}$. Finally, $[\ \_\_\_\ ]^8 \equiv_q 16$.

Give an example of a 3 digit prime $p :=$ ____ with $2 \in \mathrm{NQR}_p$, and values $r :=$ ____ and $s :=$ ____ satisfying $r^2 \equiv_p -1$ and $s^2 \equiv_p r$.

End of Individual Optional Project-Z

Please PRINT your name and ordinal. Ta:

____________________ Ord: ____

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: ______________________

Folks, I have had a great time working with you this Semester. Stop by next semester to "Talk Math".

Cheers, Prof. Sieve-brain