Staple!

Sets and Logic
MHF3202 8768

Optional Individual-Project-Y

Prof. JLF King
31Aug2015

Due no-later-than: noon, Friday, 25Apr2014 slid *completely* under my office door, LITTLE HALL 402 [top floor, north-east corner.] Then please *email me* that you have handed-in a project. Please write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

Y1: *Show no work.*

a Between sets $\mathbf{X} := \mathbb{Z}_+$ and $\mathbf{Y} := \mathbb{N}$, consider injections $f{:}\mathbf{X}\hookrightarrow\mathbf{Y}$ and $h{:}\mathbf{Y}\hookrightarrow\mathbf{X}$, defined by

$$f(x) := 3x \quad \text{and} \quad h(y) := y + 5\,.$$

Schröder-Bernstein produces a set $G \subset h(\mathbf{Y}) \subset \mathbf{X}$ st., letting $U := \mathbf{X} \smallsetminus G$, the fnc $\beta{:}\mathbf{X}\hookrightarrow\!\!\!\!\rightarrow\mathbf{Y}$ is a *bijection*, where

$$*: \qquad \beta\lfloor_U := f\lfloor_U \quad \text{and} \quad \beta\lfloor_G := h^{-1}\lfloor_G\,.$$

For this (f, h), the (G, U) pair is unique. Computing,
$\beta(56)=$ ________. $\beta(137)=$ ________. $\beta^{-1}(603)=$ ________

b Each three sets Ω, B, C engender a natural bijection, $\Theta{:}\Omega^{B\times C}\hookrightarrow\!\!\!\!\rightarrow[\Omega^B]^C$, defined, for each $f \in \Omega^{B\times C}$, by

$\Theta(f) := \Big[c \mapsto [$ ________________________________ $]\Big]$.

Its inverse-map $\Upsilon{:}[\Omega^B]^C\hookrightarrow\!\!\!\!\rightarrow\Omega^{B\times C}$ has, for $g \in [\Omega^B]^C$,
$\Upsilon(g) := \Big[(b, c) \mapsto [$ ________________________________ $]\Big]$.

Your 2 essay(s) must be TYPESET, *and Double or Triple spaced. Use the* Print/Revise *cycle to produce good, well thought out, essays. Start each essay on a* ***new*** *sheet of paper. Do* **not** *restate the problem; just solve it.*

Y2: [A dodecahedron is a convex polyhedron having 12 faces, 20 vertices and 30 edges; the faces are pentagons.] Two vertices of a regular dodecahedron are ***neighbors*** if they are distinct vertices of a common pentagon. [Each vertex has $[3\cdot 4] - 3 = 9$ neighbors.] Write $v \sim w$ to indicate that v and w are neighbors. Easily, $\sim$ is symmetric, and anti-reflexive. You can check that $\sim$ is not transitive.

A ***labeling*** of a regular dodecahedron assigns, to each vertex, a *positive integer*. A labeling is ***legal*** IFF *no* pair $v \sim w$ of vertices is assigned the same label.

i Prove there is no legal labeling with vertex sum [the sum of the 20 labels] equaling 59.

ii Let $\mathcal{S} \subset \mathbb{Z}_+$ be the *set* of sums obtainable from legal-labelings. Characterize, with proof, $\mathcal{S}$; you will likely need to construct some particular legal-labelings. [You showed, above, that $\mathcal{S} \not\ni 59$.]

Y3: [For free: Union Thm: *A countable union of countable-sets is countable.* Also, Finite-subset Thm: *The collection of **finite** subsets of a countable set, is countable.* If needed, use $\mathcal{P}_{\text{Fin}}(S)$ for the collection of *finite* subsets of a set S, and use $\mathcal{P}_\infty(S)$ for the collection of **infinite** subsets of S.] Below, a ***blip*** is an *infinite* set of positive integers. A ***family*** $\{B_i\}_{i \in \mathbf{J}}$ is a set of *distinct* blips, i.e, $\forall i,k \in \mathbf{J}$: $[i \neq k] \Rightarrow [B_i \neq B_k]$.

a Suppose $\forall i,k \in \mathbf{J}$ that $[i \neq k] \Rightarrow [B_i \cap B_k = \varnothing]$. Construct, with proof, an *injection* $g{:}\mathbf{J} \hookrightarrow \mathbb{N}$, to conclude that this index-set $\mathbf{J}$ is only countable.

b Instead suppose $|B_i \cap B_k| \leqslant 1$, for each distinct index-pair $i, k \in \mathbf{J}$. Prove that $\mathbf{J}$ is only countable. Once solved, weaken the hypothesis to $|B_i \cap B_k| \leqslant 2$, yet still show $\mathbf{J}$ countable. Finally, weaken to $|B_i \cap B_k| \leqslant 3$, and prove that $\mathbf{J}$ is only countable. Can you generalize?

c [Challenging/Creative; Making $\mathbf{J}$ equal $\mathbb{R}$.] Construct a *specific* family $\{B_x\}_{x \in \mathbb{R}}$, that is, define a *specific* injection $h{:}\mathbb{R} \hookrightarrow \mathcal{P}_\infty(\mathbb{Z}_+)$, so that: *For each distinct index-pair $x, y \in \mathbb{R}$, the intersection $[B_x \cap B_y]$ is finite.*

End of *Optional* Individual-Project-Y

Y1:	___ ___	60pts
Y2:	___ ___	85pts
Y3:	___ ___ ___	115pts
Poorly stapled, or not double/triple-spaced, or font too small. :	___ ___	−30pts
Total:	___ ___ ___	260pts

Please PRINT your name *and* ordinal. *Ta:*

Ord:

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature:

Folks, I have had a great time working with you this Semester. Stop by next semester to "Talk Math".

Cheers, Prof. Sieve-brain