

**Y1:** Show no work.

**z** If  $\lim_{x \rightarrow 0^+} 8/x$  equals  $\infty$ , then  $\lim_{x \rightarrow 0^+} 5/x$  is Circle:  
 Prof. King's beret                      Brine                      ↻

**a** U.F.  $x = x(t)$  satisfies  $2x^{(3)} + 5x^{(2)} - x = 0$ .  
 Then  $Y := \begin{bmatrix} x \\ x' \\ x'' \end{bmatrix}$  satisfies  $Y' = M \cdot Y$ , where M is  
 this 3x3 matrix of numbers: 


.

**b** Fncs  $x(t)$  and  $y(t)$  satisfy this system of DEs,

$$\begin{aligned} x' + 4x - y &= 0, \\ y' + 2x + 7y &= 0. \end{aligned}$$

It can be written as  $Z' = R \cdot Z$ ,  
 where  $Z := \begin{bmatrix} x \\ y \end{bmatrix}$  and R is matrix ......

Characteristic poly of R is  $\phi_R(z) =$  .....  
 A soln has  $x(t)$  a linear combination of  $e^{\alpha t}$  and  $e^{\beta t}$   
 for numbers  $\alpha =$  ..... and  $\beta =$  ......

**c** Fnc  $y_{\alpha, \beta}(t) = \alpha e^{At} + \beta e^{Bt} + P \cdot \sin(t) + Q \cdot \cos(t)$   
 is the general soln to

\*:  $3y'' + 4y' + y = \cos(t),$   
 with numbers  $A =$  .....,  $B =$  .....,  $P =$  .....,  $Q =$  ......

**d** Matrices B, U, N are 3x3, with B invertible and N nilpotent. [Use I for the 3x3 identity matrix.]

- |  |               |
|--|---------------|
| Matrix $BNB^{-1}$ is nilpotent:                  | AT   AF   Nei |
| Each entry of $e^{tN}$ is a polynomial:          | AT   AF   Nei |
| Matrix $e^N$ is nilpotent:                       | AT   AF   Nei |
| $N^2$ is the zero-matrix:                        | AT   AF   Nei |
| Matrix $e^{[U+I]U}$ equals $e^U \cdot e^{U^2}$ : | AT   AF   Nei |
| Matrix $e^{[U^2]}$ equals $[e^U]^2$ :            | AT   AF   Nei |

**Y2:** Show no work.

**e** Consider linear DiffOp

$$V(y) := ty'' - [1+t]y' + y.$$

Verify [for yourself] that  $V(Y_0) = 0$  and  $V(Y_1) = 0$ , where  $Y_0 := e^t$  and  $Y_1 := 1+t$ . Their Wronskian is  $W(Y_0, Y_1) = \dots$ . Then VoP tells us that

$y_{\alpha,\beta} := \dots$  is the general soln to  $V(y_{\alpha,\beta}) = 3t^2$ .

**Remark.** To save typing, define abbrevs  $\mathcal{E} := e^t$ , and [using R for Reciprocal]  $\mathcal{R} := e^{-t}$ . Thus

$$\mathcal{E}' = \mathcal{E}, \quad \mathcal{R}' = -\mathcal{R}, \quad \text{and} \quad \mathcal{E} \cdot \mathcal{R} = 1. \quad \square$$

**VoP soln.** Note  $Y_0' = \mathcal{E}' = \mathcal{E}$  and  $Y_1' = 1$ . Hence

$$D := W(Y_0, Y_1) = \mathcal{E} \cdot 1 - \mathcal{E} \cdot [1+t] \stackrel{\text{note}}{=} -t\mathcal{E}.$$

VoP needs a *monic* operator, so define

$$L(y) := \frac{1}{t}V(y) \stackrel{\text{note}}{=} y'' - [\frac{1}{t} + 1]y' + \frac{1}{t}y.$$

We seek a particular soln  $\mathbf{y}$  to  $V(\mathbf{y}) = 3t^2$ . It suffices to find a fnc  $\mathbf{s}$  so that  $V(\mathbf{s}) = t^2$ , for then  $V(3\mathbf{s}) = 3t^2$ , since  $V$  is linear. We'll then set  $\mathbf{y} := 3\mathbf{s}$ .

Dividing  $V(\mathbf{s}) = t^2$  by  $t$ , then, we seek an  $\mathbf{s}$  satisfying  $L(\mathbf{s}) = t$ ; our target function is  $G = G(t) := t$ .

**Applying VoP.** We write

$$*: \quad \mathbf{s} := f_0 \cdot Y_0 + f_1 \cdot Y_1. \quad \text{Let } h_j := f'_j.$$

Recall that

$$h_0 = -Y_1 \cdot \frac{G}{D} \quad \text{and} \quad h_1 = Y_0 \cdot \frac{G}{D}. \quad \text{Then} \\ y_{\alpha,\beta} = \mathbf{y} + [\alpha Y_0 + \beta Y_1].$$

Computing,

$$\frac{G}{D} = \frac{t}{-t\mathcal{E}} \stackrel{\text{note}}{=} \frac{-1}{\mathcal{E}} = -\mathcal{R}.$$

Hence

$$h_0 = -[1+t] \cdot [-\mathcal{R}] \stackrel{\text{note}}{=} \mathcal{R} + t\mathcal{R}, \quad \text{and} \\ h_1 = \mathcal{E} \cdot \frac{-1}{\mathcal{E}} \stackrel{\text{note}}{=} -1.$$

Anti-diffing,

$$f_0 = \int h_0 = -[2+t]\mathcal{R}, \quad \text{and} \\ f_1 = \int h_1 = -t.$$

Consequently,

$$f_0 \cdot Y_0 = -[2+t]\mathcal{R} \cdot \mathcal{E} \stackrel{\text{note}}{=} -[2+t].$$

And  $f_1 \cdot Y_1 = -t \cdot [1+t] \stackrel{\text{note}}{=} -[t+t^2].$

Adding,  $\mathbf{s} = -[2+t+t^2] = -[2+2t+t^2].$

Recall that  $V$  sends  $Y_1 \stackrel{\text{note}}{=} [1+t]$  to zero. So we can **redefine**  $\mathbf{s}$  by adding  $2Y_1$  to it. This yields a simpler

$$\mathbf{s} := -[t^2].$$

So  $\mathbf{y} = 3\mathbf{s} = -3t^2$ . Thus

$$y_{\alpha,\beta} = -3t^2 + \alpha e^t + \beta[1+t].$$

**Checking.** We calculate that  $V(\mathbf{s})$  equals

$$t \cdot \mathbf{s}'' - [1+t] \cdot \mathbf{s}' + \mathbf{s} \\ = -[t \cdot 2 - [1+t] \cdot 2t + t^2] = -[-2t^2 + t^2] \stackrel{\text{note}}{=} t^2,$$

as desired.

**f** With  $f(x) := e^{7x}$  and  $g(x) := e^{4x}$ , then

$$[f \otimes g](5) = \dots$$

With  $\mathbf{1}()$  the constant-1 fnc and  $F(x) := \sin(5x)$ , then, convolution

$$[\mathbf{1} \otimes F](x) = \dots$$

End of Y-Class

**Y1:** \_\_\_\_\_ 140pts

**Y2:** \_\_\_\_\_ 95pts

**Total:** \_\_\_\_\_ 235pts