Staple!

Ord: ___ ___

Plex
MAA4402 8436

Class-X

Prof. JLF King
Wednesday, 17Nov2021

***Read!* the Notation!** Use $\mathsf{C}_r := \mathrm{Sph}_r(0)$.

The expression FUNCTION $= \mathsf{u} + \mathbf{i}\mathsf{v}$ means that u is the real-part of FUNCTION, and v is the imaginary-part.

$\mathrm{Res}(\beta, \mathbf{q})$ is the residue of *function* $\beta()$ at the point $\mathbf{q}$. Please simplify answers, when possible.

X1: Short answer. Show no work.

a) 20 Which action *loses pts*? (circle) *Writing-in-sentences* *Writing-t-different-from-+* *Tiny - tiny - writing* *Writing-LARGE*

b) 10 25 Fnc $\alpha(z) := \dfrac{\cosh(7z)}{[z-2]^2\, z^2}$ has a $z{=}0$ pole of order 2.

And $\mathrm{Res}(\alpha, 0) =$ 1/4.

Efficient. As $f(z) := z^2\alpha(z) = \dfrac{\cosh(7z)}{[z-2]^2}$, the requested residue is $R := \frac{1}{1!}f'(0)$. Since $\sinh(7\cdot 0)$ is zero, I will replace that term by **0** in the derivative. Hence

$$R = \frac{\mathbf{0} - \cosh(7z)\cdot 2[z-2]}{[z-2]^4}\bigg|_{z=0} = \frac{-1\cdot -4}{[-2]^4} = 1/4. \blacklozenge$$

25 C Let $J := \int_{C_2} z^4 \sin(\frac{7}{z})\,dz$. So $J = 2\boldsymbol{\pi}\mathbf{i} \cdot \frac{7^5}{5!} = \frac{7^5 \boldsymbol{\pi}\mathbf{i}}{60}$.

Power to the Series! Plugging $\frac{7}{z}$ into the PS for $\sin()$ gives

$$\sin(\tfrac{7}{z}) = \frac{[7/z]^1}{1!} - \frac{[7/z]^3}{3!} + \frac{[7/z]^5}{5!} - \ldots$$

In product $z^4 \sin(\frac{7}{z})$, then, the coeff of z^{-1} is $7^5/5!$, which is therefore $\text{Res}(z^4 \sin(\frac{7}{z}), z{=}0)$. Hence $J = 2\boldsymbol{\pi}\mathbf{i} \cdot 7^5/5!$. ♦

10 25 d Defining $\vec{\mathbf{a}} = (a_0, a_1, \ldots)$ by $\cos(2z) = \sum_{n=0}^{\infty} a_n z^n$, coefficient $a_4 =$ $2^4/4! = 2/3$. Let

$$\sum_{n=0}^{\infty} r_n z^n = \frac{1}{\cos(2z)}$$

be the reciprocal PS. Then $r_4 =$ $10/3$.

[*Hint:* You can compute the reciprocal as we did in class.]

Soln. From *PlexNotes*, recall the reciprocal power-series

$$\begin{aligned} \mathsf{A} &= 1 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots \quad \textit{and} \\ \mathsf{R} &= 1 + \underbrace{-a_1}_{r_1} z + \underbrace{[[a_1]^2 - a_2]}_{r_2} z^2 + \ldots, \end{aligned}$$

satisfying $\mathsf{AR} = 1$. Doubling the sub/super-scripts gives

$$\begin{aligned} \mathsf{A} &= 1 + a_2 z^2 + a_4 z^4 + a_6 z^6 + \ldots \quad \textit{and} \\ \mathsf{R} &= 1 + \underbrace{-a_2}_{r_2} z^2 + \underbrace{[[a_2]^2 - a_4]}_{r_4} z^4 + \ldots, \end{aligned}$$

for new A and R, still satisfying $\mathsf{AR} = 1$. Since

$$\cos(2z) = 1 - \tfrac{2^2}{2!} z^2 + \tfrac{2^4}{4!} z^4 - \ldots = 1 - 2z^2 + \tfrac{2}{3} z^4 - \ldots$$

Thus $r_4 = [-2]^2 - \frac{2}{3} = \frac{10}{3}$. ♦

X2: Short answer. Show *no* work.

Let $f(z) := z^4 + 5z^2 + 4$. Reciprocal $H(z) := 1/f(z)$ has, in the upper half-plane, two poles $\mathbf{p}$ and $\mathbf{q}$, where $\mathbf{p}$ lies closer to the origin than $\mathbf{q}$.

So $\text{Res}(H, \mathbf{p}) =$ $1/[6\,\mathbf{i}]$ *and* $\text{Res}(H, \mathbf{q}) =$ $-1/[12\,\mathbf{i}]$.

Our D-contour technique applies to H.

Thus $J := \int_{-\infty}^{+\infty} \frac{1}{z^4 + 5z^2 + 4}\,\mathrm{d}x =$ $\boldsymbol{\pi}/6$.

Two poles. Factoring, $f(z) = [z^2+1][z^2+4]$. Thus, the upper half-plane poles are $\mathbf{p} = 1\mathbf{i}$ and $\mathbf{q} = 2\mathbf{i}$. As these are simple poles, CIF hands us

$$\text{Res}(H,\, \mathbf{p}{=}\mathbf{i}) = \frac{1}{[z^2+4][z+1\mathbf{i}]}\bigg|_{z=1\mathbf{i}} = \frac{1}{3 \cdot 2\mathbf{i}} \quad \textit{and}$$

$$\text{Res}(H,\, \mathbf{q}{=}2\mathbf{i}) = \frac{1}{[z^2+1][z+2\mathbf{i}]}\bigg|_{z=2\mathbf{i}} = \frac{1}{-3 \cdot 4\mathbf{i}}.$$

The arc-integral $\int_{\mathsf{A}_r} \frac{1}{f}$ goes to zero as $r \nearrow \infty$. Hence

$$J = 2\boldsymbol{\pi}\mathbf{i} \cdot_{\textit{Sum-of-residues}} = 2\boldsymbol{\pi}\mathbf{i} \cdot \left[\frac{1}{6\,\mathbf{i}} - \frac{1}{12\,\mathbf{i}}\right] = \frac{\boldsymbol{\pi}}{6}. \quad \blacklozenge$$

OYOP: *In grammatical English* ***sentences****, write your essay on every* 2^{nd} *line* (usually)*, so I can easily write between the lines.*

X3: α Precisely state the Gauss Mean-value thm, with all the hypotheses. State it as a *formal* theorem.

β Carefully prove the Gauss MVT, using the CIF.

Soln. See *PlexNotes*, or our textbook. ♦

X4: Consider an entire fnc $g = \mathsf{u} + \mathbf{i}\mathsf{v}$ whose imaginary-part, v is bounded, e.g, $|\mathsf{v}(z)| \le 5$ for all z. Prove that g is constant. [*Hint:* Somehow prove that g is bounded, then apply SomebodyOrOther's thm. (*Who's thm?* State the theorem formally.) You may want to construct an auxiliary function from g.]

Soln. Let $F(z) := \exp(-\mathbf{i} \cdot g(z))$. So $|F(z)| = |\mathsf{e}^{-\mathbf{i}\,\mathsf{u}(z) + \mathsf{v}(z)}|$. Thus $|F(z)| = |\mathsf{e}^{\mathsf{v}(z)}| \le \mathsf{e}^5$ since, *on the reals*, exp() is a strictly-increasing fnc.

Liouville's thm now asserts F is *constant*. Since exp, *on the reals*, is 1-to-1, it now follows that g is constant. ♦

X1:	___ ___ ___	115pts
X2:	___ ___	60pts
X3:	___ ___	55pts
X4:	___ ___	40pts
Total:	___ ___ ___	270pts

Please PRINT your name and ordinal. Ta:

Ord:

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: *Energentic Student!*