Staple!

Ord: ___ ___

Linear Algebra
MAS4105 *3249*

Individual Optional Project

Prof. JLF King
Thursday, 14Apr2022

IOP is due by **2PM, Thursday, 21Apr2022**. Essays must be TYPED, [LaTeX is recommended, not req.]

Inner-products are conjugate-linear in the first argument, and linear in the second. Use A^* for the conjugate-transpose (the adjoint) of A. Let $\mathbb{M}_K$ be the $\mathbb{C}$-VS of $K\times K$ $\mathbb{C}$-matrices; so $\text{Dim}(\mathbb{M}_K) = K^2$.

Use $\text{LinDir}(\mathbf{u},\mathbf{D}) := \{\mathbf{u}+t\mathbf{D} \mid t\in\mathcal{F}\}$ for the line through $\mathbf{u}$ in direction $\mathbf{D}$.

Write **DNE** if the object does not exist or the operation cannot be performed. NB: $\textbf{DNE} \neq \{\} \neq 0$.

X1: [i] *Essay:* For vectors, i.e, matrices, $\mathsf{A},\mathsf{B}\in\mathbb{M}_K$ define

$$\dagger: \qquad \langle \mathsf{A},\mathsf{B}\rangle := \text{Trace}(\mathsf{A}^*\mathsf{B}).$$

Prove that $\langle\cdot,\cdot\rangle$ is a $\mathbb{C}$-inner-product.

[ii] *Short answer:* Equip $\mathbb{M}_2$ with ($\dagger$)-IP. Define vectors $\mathsf{G}_1 := \begin{bmatrix}0 & \mathbf{i}\\ 0 & 1\end{bmatrix}$, $\mathsf{G}_2 := \begin{bmatrix}2 & 2\mathbf{i}\\ 0 & 0\end{bmatrix}$, and $\mathsf{G}_3 := \begin{bmatrix}3\mathbf{i} & 9\mathbf{i}\\ 6 & 15\end{bmatrix}$. On line $\mathbb{L} := \text{LinDir}(\mathsf{G}_2,\mathsf{G}_1)$, the closest point to G_3 is $\left[\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right]$.

[iii] *Short ans:* Gram-Schmidtify $(\mathsf{G}_1,\mathsf{G}_2,\mathsf{G}_3)$ to form $(\mathsf{B}_1 := \mathsf{G}_1, \mathsf{B}_2, \mathsf{B}_3)$, where $\mathsf{B}_1\perp\mathsf{B}_2$, $\mathsf{B}_1\perp\mathsf{B}_3$, $\mathsf{B}_2\perp\mathsf{B}_3$, with $\text{Spn}(\mathsf{B}_1,\mathsf{B}_2) = \text{Spn}(\mathsf{G}_1,\mathsf{G}_2)$ and $\text{Spn}(\mathsf{B}_1,\mathsf{B}_2,\mathsf{B}_3) = \text{Spn}(\mathsf{G}_1,\mathsf{G}_2,\mathsf{G}_3)$.

Then $\mathsf{B}_2 = \left[\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right]$, $\mathsf{B}_3 = \left[\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right]$.

X2: Matrices $\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}$ have sizes $K\times K$, $K\times L$, $L\times K$, $L\times L$, resp., forming block matrix $\mathsf{G} := \left[\begin{smallmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{smallmatrix}\right]$, with A *invertible.* Let $\mathbf{I}$ and $\mathbf{J}$ be the $K\times K$ and $L\times L$ identity matrices.

[a] Prove this block-matrix–formula:

$$\dagger: \quad \begin{bmatrix}\mathbf{I} & \mathbf{0}_{\text{Mat}}\\ -\mathsf{C}\mathsf{A}^{-1} & \mathbf{J}\end{bmatrix}\cdot\begin{bmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{bmatrix} = \begin{bmatrix}\mathsf{A} & \mathsf{B}\\ \mathbf{0}_{\text{Mat}} & \mathsf{D}-\mathsf{C}\mathsf{A}^{-1}\mathsf{B}\end{bmatrix}.$$

[b] Using that, in ($\dagger$), the first and third block matrices are block-*triangular*, prove that

$$\ddagger: \quad \text{Det}(\mathsf{G}) = \text{Det}(\mathsf{A})\cdot\text{Det}(\mathsf{D}-\mathsf{C}\mathsf{A}^{-1}\cdot\mathsf{B}).$$

[c] Suppose now that A and D have the same size (hence $\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}$ are each, say, $K\times K$). *If* A and C commute, prove

$$*: \quad \text{Det}(\mathsf{G}) = \text{Det}(\mathsf{AD}-\mathsf{CB}).$$

[d] Construct a CEX to ($*$) when $\mathsf{A}\not\rightleftarrows\mathsf{C}$. (Use $K := 2$.)

X3: A matrix $\mathsf{W}\in\text{MAT}_{N\times N}(\mathbb{C})$ is ***weird*** if for each complex N-colvec U, the product $\mathsf{U}^*\mathsf{W}\mathsf{U}$ is real. (*Dis*)*prove*: *If* W *is weird, then* $\mathsf{W}^* = \mathsf{W}$.

[For a *CEX*, give a *specific* N and *specific* matrix W whose entries are specific complex numbers. For a *proof*, recall the *quantifier*: $\forall\mathsf{U}$: $\mathsf{U}^*\mathsf{W}\mathsf{U}\in\mathbb{R}$.]

X4: Let $\mathbf{V}$ be the $\mathbb{R}$-VS of $K\times K$ real matrices; so $\text{Dim}(\mathbf{V}) = K^2$. Let $\mathbf{P}_K\subset\mathbf{V}$ be the span of the $K!$ many [K-factorial] permutation matrices. Let $h(K) := \text{Dim}(\mathbf{P}_K)$.

[α] Compute $h(K)$ for $K = 1,2,3$ to conjecture that $h(K) =$ ....

[β] Prove your formula, for a *posints* K. [Depending on how you proceed, this proof may be challenging. Perhaps you can give a heuristic argument, or verify your formula for more values of K. Your ideas here are more important than a completely rigorous proof.]

NAME: ....

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor or TA."*

Signature: ....