*Staple!*

**Ord:** ___ ___

Plex
MAA4402 8436

**Class-W**

Prof. JLF King
Wedn, 27Oct2021

**Welcome!** Use "$f(x)$ notation" when writing fncs; in particular, for trig and log fncs. E.g, write "$\sin(x)$" rather than the horrible $\sin x$ or $[\sin x]$.

Write unambiguously e.g, $1/a+b$ should be *bracketed* either $[1/a]+b$ or $1/[a+b]$, as appropriate. (Be careful with **negative** signs!) Write **DNE** if the object does not exist or the operation cannot be performed. NB: **DNE** $\neq \{\} \neq 0$.

**W1:** Short answer. Show no work.

a 10 Prof. King wears bifocals, and cannot read small handwriting. Circle one:
**True!** **Yes!** ***Who??***

b 15 Prof. King thinks that submitting a Robert Long prize essay [typically 2 prizes, $500 total] is a *really good idea*. A ten-page essay is fine. Date for the emailed-PDF is Sunday, March 27, 2022.

Circle: **Yes** **True** ***Résumé material!***

c 25 Define $f(x+\mathbf{i}y) := xy + \mathbf{i}x$. Let $\mathsf{L}$ be the line-segment from the origin to $2+\mathbf{i}$. Then $\int_{\mathsf{L}} f(z)\,dz =$ ______________ .

**Soln:** With "velocity" $V := 2+\mathbf{i}$, parametrize $\mathsf{L}$ by $z(t) := V\cdot t$. So $x = 2t$ and $y = t$, and $z'(t) = V$. Thus $\int_{\mathsf{L}} f$ equals

$$\int_0^1 \Big[2t^2 + \mathbf{i}\cdot 2t\Big] V\,dt \;=\; V\cdot\Big[\tfrac{2}{3} + \mathbf{i}\Big] \;=\; \tfrac{1}{3}[1 + 8\mathbf{i}].$$

d 30 Let $\mathsf{C}$ be SCC $\mathrm{Sph}_3(\mathbf{i})$, a circle of radius 3. Value $J := \oint_{\mathsf{C}} \frac{e^{3z}}{[z-2]^5}\,dz =$ ______________

[Answer may be written as a product, using powers and factorials.]

**Soln:** With suitable hypotheses, GCIF concludes with

$$\oint_C \frac{h(z)}{[z-w]^{n+1}}\,dz \;=\; \frac{2\pi\mathbf{i}}{n!}\cdot h^{(n)}(w).$$

Since $2 \in \mathring{\mathsf{C}}$, and $h(z) := e^{3z}$ is holomorphic, we may apply GCIF with $w := 2$ and $n := 4$. Note $h^{(4)}(z) = 3^4\,e^{3z}$. Thus $h^{(4)}(2) = 3^4\,e^6$. Consequently

$$J \;=\; \frac{2\pi\mathbf{i}}{4!}\cdot 3^4\,e^6 \;=\; \frac{27}{4}\cdot\pi\cdot e^6\cdot\mathbf{i}.$$

30 e Fnc $u(x,y) := 2xy + x$ has harmonic conjugate $v(x,y) =$ $y^2 - x^2 + y$ .

*Soln.* Cauchy-Riemann gives $v_y = u_x = 2y + 1$. Anti-diffing gives

$$v = y^2 + y + \mathrm{Fnc}(x).$$

As $v_x \overset{\text{C-R}}{=\!=} -u_y = -2x$, anti-diffing produces

$$v = -x^2 + \mathrm{Fnc}(y).$$

Reconciling gives $v = y^2 - x^2 + y$ $_{+\,\text{Number}}$.

The holomorphic fnc $f := u + \mathbf{i}v$ is $f(z) := z - \mathbf{i}z^2$. ♦

**W2:** Short answer; fill-in the 5 blanks.

Consider the logarithmic spiral, S, turning CCW as it spirals out to $\infty$, crossing the $\mathbb{R}$-axis at points $\{3^n\}_{n\in\mathbb{Z}}$.

i Similar to what we did in class, naturally parametrize S by $\sigma:\mathbb{R}\to\mathbb{C}$ of form $\sigma(t) = e^{M\cdot t}$,

where $M = \frac{\ln(3)}{2\pi} + \mathbf{i}$, with $\sigma(-2\pi) = \frac{1}{3}$, $\sigma(0) = 1$, $\sigma(2\pi) = 3, \ldots, \sigma(2\pi\cdot n) = 3^n$.

ii Let **A** be the *Arclength* of one turn of the spiral, starting at $3\in\mathbb{R}$, ending at $9\in\mathbb{R}$. Set up the specific integral $\mathbf{A} = \int_\alpha^\beta f(t)\,dt$, where

$f(t) = |\sigma'(t)| = |M|\cdot\exp\left(\frac{\ln(3)}{2\pi}\cdot t\right)$,

with $\alpha = 2\pi$ and $\beta = 4\pi$.

iii Computing the integral gives number $\mathbf{A} = \frac{\sqrt{\ln(3)^2 + [2\pi]^2}}{\ln(3)}\cdot\left[9 - 3\right]$.

*Predictions.* Generalize the scale, 3, to $V\geq 1$. What do we foresee, as $V\nearrow\infty$ or $V\searrow 1$?

Times $t_1 < t_2$ determine points $\mathbf{p}_j := \sigma(t_j)$ on the spiral, at radial distances $r_j$. As scale $V\nearrow\infty$, the spiral from $\mathbf{p}_1$ to $\mathbf{p}_2$ looks more-and-more like a line segment, yielding prediction

$$\dagger:\quad \lim_{V\nearrow\infty}\frac{\text{Arclen}_V(\mathbf{p}_2, \mathbf{p}_1)}{r_2 - r_1} = 1.$$

In the other direction, sending $V\searrow 1$ turns the spiral into circling on the unit-circle. Since our time parameter, $t$, is angle, we should expect

$$\ddagger:\quad \lim_{V\searrow 0}\text{Arclen}_V(\mathbf{p}_2, \mathbf{p}_1) = t_2 - t_1.$$

□

*Spiraling in control.* Our desired parametrization is

$$\sigma(t) := \left[e^{\ln(V)\cdot\frac{t}{2\pi}}\right]\cdot e^{\mathbf{i}t} = \exp(Mt),$$

where $L := \frac{\ln(V)}{2\pi}$ and $M := L + \mathbf{i}$.

Note $\sigma'(t) = M\cdot e^{Mt}$. Setting $\alpha = 2\pi$ and $\beta = 4\pi$, then, the requested arclength, **A**, is

$$\int_\alpha^\beta |\sigma'(t)|\,dt = |M|\cdot\int_\alpha^\beta|\exp(Mt)|\,dt$$
$$= |M|\cdot\int_\alpha^\beta\exp(L\cdot t)\,dt$$
$$*:\quad = \frac{|M|}{L}\cdot\exp(L\cdot t)\Big|_{t=\alpha}^{t=\beta}.$$

By algebra, $\frac{|M|}{L} = \frac{\sqrt{L^2+1^2}}{L}\cdot\frac{2\pi}{2\pi} = \frac{\sqrt{\ln(V)^2+[2\pi]^2}}{\ln(V)}$. In our particular case, $\mathbf{A} = \frac{\sqrt{\ln(3)^2+[2\pi]^2}}{\ln(3)}\cdot\left[9 - 3\right]$. ♦

*Analysis.* Observe that $\exp(L\cdot t)$ is the radial distance of $\sigma(t)$ [from the origin]. At times $t_1 < t_2$, write the distances as $r_1 < r_2$. So the spiral arclength from $\sigma(t_1)$ to $\sigma(t_2)$ is

$$\mathbf{A}_V := \frac{\sqrt{\ln(V)^2+[2\pi]^2}}{\ln(V)}\cdot\left[r_2 - r_1\right].$$

Prediction (†) immediately holds for $\mathbf{A}_V$. For (‡), we need l'Hôpital's rule. It is more convenient to us our earlier (∗)-formula: Our $\mathbf{A}_V$ equals

$$\frac{|M|}{L}\cdot\exp(L\cdot t)\Big|_{t=t_1}^{t=t_2} \stackrel{\text{note}}{=\!=} |M|\cdot\frac{\exp(L\cdot t_2) - \exp(L\cdot t_1)}{L}.$$

As $V\searrow 1$, note $L\searrow 0$. And $\lim_{L\searrow 0}|M| = 1$. Thus $\lim_{V\searrow 1}\mathbf{A}_V$ equals

$$\lim_{L\searrow 0}\frac{e^{Lt_2} - e^{Lt_1}}{L} \stackrel{\text{l'Hôpital's}}{=\!=\!=} \lim_{L\searrow 0}\frac{t_2\,e^{Lt_2} - t_1\,e^{Lt_1}}{1}$$

which indeed equals $t_2 - t_1$. □

OYOP: *In grammatical English* ***sentences****, write your essay on every* $2^{\text{nd}}$ *line* (usually)*, so I can easily write between the lines.*

**W3:** Below, $h:\mathbb{C}\to\mathbb{C}$, and $\mathsf{S}\subset\mathbb{C}$ is a closed-curve, and $\mathbf{w}\in\mathbb{C}$ is an *appropriate* point.

$\alpha$ Detailing the precise conditions needed on $h$, $\mathsf{S}$ and $\mathbf{w}$, *carefully* state the Cauchy Integral Formula Theorem.

$\beta$ Recall the Cauchy Homotopy Thm: *Suppose closed-curves* $\mathsf{S}$ *and* $\mathsf{R}$ *are homotopic in an open set on which a fnc* $f$ *is holomorphic. Then* $\oint_{\mathsf{S}} f = \oint_{\mathsf{R}} f$.

Use the above CHT to give a formal proof of the Cauchy Integral Formula Theorem. Also draw LARGE pictures showing the ideas in the proof.

*Soln.* See PLEXNOTES, ≈P.22, or FIRST COURSE... or *Brown & Churchill.* ♦

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|---|---|---|
| **W1:** | ___ ___ ___ | 110pts |
| **W2:** | ___ ___ | 65pts |
| **W3:** | ___ ___ | 85pts |
| **Total:** | ___ ___ ___ | 260pts |

*Please PRINT your name and ordinal. Ta:*

Ord:

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: