Staple!

Ord: ___ ___

Plex
MAA4402 8436

Bonus-B

Prof. JLF King
Tuesday, 02Nov2021

Hola. This bonus is due at BoC, Monday, 08Nov2021. Your essay(s) must be TYPED, and (if possible) Double spaced. Use the *Print/Revise* cycle to produce good, well thought out, essays. Start each essay on a *new* sheet of paper. Print out this Bonus sheet; it is the first page of your stapled write-up.

Use Δ for the Laplacian operator. Use *PlexNotes.* notation.

The essays must be Use "$f(x)$ notation" when writing fncs; in particular, for trig and log fncs. E.g, write "$\sin(x)$" rather than the horrible $\sin x$ or $[\sin x]$.

Write unambiguously e.g, $1/a+b$ should be *bracketed* either $[1/a]+b$ or $1/[a+b]$, as appropriate. (Be careful with **negative** signs!). Use alignment in displays.

B0: Suppose f has an ***isolated singularity*** at $\mathbf{q} \in \mathbb{C}$; that is, for some $r>0$ our f is holomorphic on $\mathrm{PBal}_r(\mathbf{q})$.

A ***residue*** of f at an isolated singularity $\mathbf{q}$ is a complex number $\mathcal{R}$ such that function

$$*: \qquad z \mapsto f(z) - \frac{\mathcal{R}}{z-\mathbf{q}}$$

has an antiderivative in some punctured ball $\mathrm{PBal}_\varepsilon(\mathbf{q})$, with $\varepsilon>0$. Prove: *If numbers $\mathcal{A}$ and $\mathcal{B}$ are residues of f at $\mathbf{q}$, then $\mathcal{A}=\mathcal{B}$.*

Soln. Let $\Gamma_{\mathcal{R}}$ denote the $(*)$ fnc. Since $\Gamma_{\mathcal{A}}$ and $\Gamma_{\mathcal{B}}$ have antiderivs, each has the Zero-loop property.

Take an $r>0$ small enough that $\mathsf{C} := \mathrm{Sph}_r(\mathbf{q})$ is inside *both* punctured-balls, defining $\mathcal{A}$ and defining $\mathcal{B}$. Thus

$$0-0 = \Big[\oint_{\mathsf{C}} \Gamma_{\mathcal{B}}\Big] - \Big[\oint_{\mathsf{C}} \Gamma_{\mathcal{A}}\Big] = \oint_{\mathsf{C}} \Big[\Gamma_{\mathcal{B}} - \Gamma_{\mathcal{A}}\Big]. \textit{ Canceling,}$$

$$0 = \oint_{\mathsf{C}} \Big[\frac{\mathcal{A}}{z-\mathbf{q}} - \frac{\mathcal{B}}{z-\mathbf{q}}\Big]\,\mathrm{d}z = [\mathcal{A}-\mathcal{B}]\cdot\oint_{\mathsf{C}} \frac{1}{z-\mathbf{q}}\,\mathrm{d}z$$

which equals $[\mathcal{A}-\mathcal{B}]\cdot 2\boldsymbol{\pi}\mathbf{i}$. Hence $\mathcal{A}=\mathcal{B}$. ♦

B1: On $\mathbf{P} := \mathrm{PBal}_\infty(0)$, the Punctured plane, functions $f(z) := \ln(|z|)$ and $g(z) := \mathrm{Arg}(z)$ are well-defined.

a) What is the largest set $\mathsf{F} \subset \mathbf{P}$ on which f is harmonic? Give a computational proof. Can you also give an abstract proof that avoids computation?

b) What is the largest set $\mathsf{G} \subset \mathbf{P}$ on which g is harmonic? Give a computational proof. Can you also give an abstract proof that avoids computation?

Pf f,g, abstract. On $\mathbf{P}$, fnc Log has real and imaginary parts $\ln(|z|)$ and $\mathrm{Arg}(z)$. Locally, Log is holomorphic except for the jump-discty of Arg on the negative real-axis.

Consequently, f is harmonic on *all* of $\mathbf{P}$ [the Log-Abs lemma] and g is harmonic on sLit-plane $\mathbf{L} := \mathbf{P} \smallsetminus \mathbb{R}_-$. ♦

Pf f, direct. Let $S := x^2 + y^2$. Then $f(x,y) = \ln(S^{1/2})$, giving

$$f_x = \frac{1}{S^{1/2}} \cdot \frac{1/2}{S^{1/2}} \cdot 2x \stackrel{\text{note}}{=\!=} \frac{x}{S}. \quad \textit{Thus,}$$

$$f_{xx} = \frac{1 \cdot S - x \cdot 2x}{S^2} = \frac{y^2 - x^2}{S^2}.$$

Symmetry gives $f_{yy} = \frac{x^2 - y^2}{S^2}$, so $f_{xx} + f_{yy} = 0$. ♦

Pf g, direct. [*Argument is slightly annoying due to the arbitrary choices,* $\arctan()$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $\mathrm{Arg}()$ on $(-\pi, \pi]$, *in the relevant defns.*]

For $x > 0$, our $g(z) = \arctan(y/x)$. The Chain rule says

$$g_y = \frac{1}{1 + [y/x]^2} \cdot \frac{1}{x} \stackrel{\text{note}}{=\!=} \frac{x}{S} \quad \textit{and}$$

$$g_x = \frac{1}{1 + [y/x]^2} \cdot \frac{-y}{x^2} \stackrel{\text{note}}{=\!=} -\frac{y}{S}. \quad \textit{Diff'ing again,}$$

$$g_{xx} = [-y] \cdot \frac{-1}{S^2} \cdot \stackrel{\text{note}}{=\!=} \frac{2xy}{S^2}.$$

After negation, g_y is symmetric with g_x. Consequently, $g_{yy} = -\frac{2xy}{S^2}$, showing g to be harmonic.

For x negative, $g(z) = -\arctan(y/x)$ and the negative of harmonic is harmonic. This leaves the $x{=}0$ annoyance.

A MORE ELEGANT APPROACH uses that rotating/scaling a harmonic fnc leaves it harmonic. Evidently, $\mathrm{Arg}(z) \in (-\pi, 0)$ implies $\mathrm{Arg}(\mathbf{i}z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence

$$\mathrm{Arg}(z) \stackrel{\text{note}}{=\!=} \mathrm{Arg}(\mathbf{i}z) - \frac{\pi}{2}$$

is harmonic. Using $\mathrm{Arg}(-\mathbf{i}z)$ similarly shows $\mathrm{Arg}(z)$ harmonic on $(0, \pi)$. ♦

B2: H-H Lemma. *Consider $\mathbb{R} \xleftarrow{h} U \xleftarrow{f} V$, where open sets $U, V \subset \mathbb{C}$. If h is harmonic and f holomorphic, then $h \circ f$ is harmonic.*

There is an abstract proof; you'll need to argue *carefully*.

There is a tedious computational proof; you'll want the C-R eqns. While not difficult, the computation will require *careful* organization and typesetting.

Pf abstract. Locally, h has an harmonic conjugate, $\widetilde{h}$, making $H := h + \mathbf{i}\widetilde{h}$ holomorphic. Hence $H \circ f$ is holomorphic (locally). Thus $\mathrm{Re} \circ H \circ f \stackrel{\text{note}}{=\!=} h \circ f$ is harmonic. ♦

Pf computation. Let $f := \mathsf{u} + \mathbf{i}\mathsf{v}$ and $G := h \circ f$, i.e,

$$G(x,y) \quad := \quad h\Big(\mathsf{u}(x,y),\, \mathsf{v}(x,y)\Big)\,.$$

Then

$$G_x = h_{\mathsf{u}} \cdot \mathsf{u}_x \quad + \quad h_{\mathsf{v}} \cdot \mathsf{v}_x\,. \qquad \textit{Thus}$$

$$\begin{aligned} G_{xx} = {} & [h_{\mathsf{uu}}\mathsf{u}_x + h_{\mathsf{uv}}\mathsf{v}_x]\cdot\mathsf{u}_x + h_{\mathsf{u}}\cdot\mathsf{u}_{xx} \\ & + [h_{\mathsf{vu}}\mathsf{u}_x + h_{\mathsf{vv}}\mathsf{v}_x]\cdot\mathsf{v}_x + h_{\mathsf{v}}\cdot\mathsf{v}_{xx}\,. \quad \textit{Similarly,} \end{aligned}$$

$$\begin{aligned} G_{yy} = {} & [h_{\mathsf{uu}}\mathsf{u}_y + h_{\mathsf{uv}}\mathsf{v}_y]\cdot\mathsf{u}_y + h_{\mathsf{u}}\cdot\mathsf{u}_{yy} \\ & + [h_{\mathsf{vu}}\mathsf{u}_y + h_{\mathsf{vv}}\mathsf{v}_y]\cdot\mathsf{v}_y + h_{\mathsf{v}}\cdot\mathsf{v}_{yy}\,. \end{aligned}$$

Sum $h_{\mathsf{u}}\cdot\mathsf{u}_{xx} + h_{\mathsf{u}}\cdot\mathsf{u}_{yy} = h_{\mathsf{u}}\cdot[\mathsf{u}_{yy} + \mathsf{u}_{xx}] = 0$, since u is harmonic. And v is harmonic, so $G_{xx} + G_{yy}$ equals

$$\begin{aligned} & h_{\mathsf{uu}}\mathsf{u}_x\mathsf{u}_x + h_{\mathsf{uv}}\mathsf{v}_x\mathsf{u}_x + h_{\mathsf{vu}}\mathsf{u}_x\mathsf{v}_x + h_{\mathsf{vv}}\mathsf{v}_x\mathsf{v}_x \\ + \; & h_{\mathsf{uu}}\mathsf{u}_y\mathsf{u}_y + h_{\mathsf{uv}}\mathsf{v}_y\mathsf{u}_y + h_{\mathsf{vu}}\mathsf{u}_y\mathsf{v}_y + h_{\mathsf{vv}}\mathsf{v}_y\mathsf{v}_y\,. \end{aligned}$$

Our C-R eqns permit replacing each v_y by u_x, and each v_x by $[-\mathsf{u}_y]$. The rewritten ΔG is

$$\begin{aligned} & h_{\mathsf{uu}}\mathsf{u}_x\mathsf{u}_x + h_{\mathsf{uv}}[-\mathsf{u}_y]\mathsf{u}_x + h_{\mathsf{vu}}\mathsf{u}_x[-\mathsf{u}_y] + h_{\mathsf{vv}}[-\mathsf{u}_y][-\mathsf{u}_y] \\ + \; & h_{\mathsf{uu}}\mathsf{u}_y\mathsf{u}_y + h_{\mathsf{uv}}\mathsf{u}_x\mathsf{u}_y + h_{\mathsf{vu}}\mathsf{u}_y\mathsf{u}_x + h_{\mathsf{vv}}\mathsf{u}_x\mathsf{u}_x \\ = \; & h_{\mathsf{uu}}\mathsf{u}_x\mathsf{u}_x + \mathbf{0} + \mathbf{0} + h_{\mathsf{vv}}\mathsf{u}_y\mathsf{u}_y \\ + \; & h_{\mathsf{uu}}\mathsf{u}_y\mathsf{u}_y + \mathbf{0} + \mathbf{0} + h_{\mathsf{vv}}\mathsf{u}_x\mathsf{u}_x \\ = \; & \big[h_{\mathsf{uu}} + h_{\mathsf{vv}}\big] \cdot \big[[\mathsf{u}_x]^2 + [\mathsf{v}_x]^2\big]\,. \end{aligned}$$

The first (leftmost) bracket is zero, since h is harmonic. ♦

B3: H-H-H failure. *Produce three harmonic functions,* h,u,v$:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, *such that*

$$G(x,y) \quad := \quad h\Big(\mathsf{u}(x,y),\, \mathsf{v}(x,y)\Big)$$

fails to be harmonic.

Soln. Function $h(\mathsf{u},\mathsf{v}) := \mathsf{u}^2 - \mathsf{v}^2$ is harmonic, as are $\mathsf{u} := x$ and $\mathsf{v} := 2y$. Yet $G(x,y) = x^2 - 4y^2$ fails, as Laplacian $\Delta(G) = 2 - 8 \neq 0$. ♦

B4: Fnc φ is holomorphic and never zero on $\mathbb{B} := \mathrm{Bal}_1(0)$. Prove that

$$\dagger\textbf{:}\qquad \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln\Bigl(\bigl|\varphi(r\mathbf{e}^{\mathbf{i}\theta})\bigr|\Bigr)\,\mathrm{d}\theta\right) \;=\; \bigl|\varphi(0)\bigr|\,,$$

for each positive $r<1$. [*Hint:* Eventually apply Gauss/Harmonic-fnc MVT. You may use previous results on this Bonus.]

Soln. Integrating around circle $\mathsf{C}_r := \mathrm{Sph}_r(0)$, the above integral, J, rewrites as

$$\oint_{\mathsf{C}_r} \ln\Bigl(\bigl|f(z)\bigr|\Bigr)\cdot\overbrace{\tfrac{|\mathrm{d}z|}{r}}^{\mathrm{d}\theta} \;=\; \tfrac{1}{r}\oint_{\mathsf{C}_r} \ln\Bigl(\bigl|f(z)\bigr|\Bigr)\,|\mathrm{d}z|\,.$$

Our Log-Abs lemma says that $h(w) := \ln(|w|)$ is harmonic on $\mathrm{PBal}_\infty(0)$. So (†) restates as

$$\ddagger\textbf{:}\qquad \frac{1}{2\pi}\cdot\frac{1}{r}\oint_{\mathsf{C}_r} h\Bigl(f(z)\Bigr)\,|\mathrm{d}z| \quad\stackrel{?}{=}\quad h\bigl(f(0)\bigr)\,.$$

Our H-H Lemma asserts that $h\circ f$ is harmonic. Since $\mathrm{Len}(\mathsf{C}_r) = 2\pi\,r$, the Harmonic Mean-Value thm applies to $h\circ f$, yielding (‡). ♦

B5: Consider an entire f which is *not* bounded. Prove that $\text{Range}(f)$ intersects each open-ball in $\mathbb{C}$.

[A suggestion is to start your proof with a phrase like: *"Suppose, FTSOContradiction, there is a ball* $\mathbf{B} := \text{Bal}_R(\mathbf{q})$ *disjoint from* $\text{Range}(f)$*, where* $\mathbf{q}\in\mathbb{C}$ *and* $R>0$*."* Arrive at a contradiction by somehow applying Liouville's thm.]

Soln. FTSOC, suppose $\text{Range}(f)$ misses some $\text{Bal}_R(\mathbf{q})$. Function $g(z) := \frac{1}{f(z) - \mathbf{q}}$ is entire and $|g(z)| \le \frac{1}{R}$. Happily, Liouville's thm now asserts g is constant; ditto for f. ♦

Please PRINT your name and ordinal. Ta:

Ord: ....

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: *Energentic Student!* ....