Volume of the $\mathfrak{D}$-ball and the Gamma function

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ABSTRACT: (See "Joe's Circle Puzzle" for an application of this, in PROBLEMS NB)

We are born knowing that... For $\mathfrak{D} \in \mathbb{Z}_+$, the $[\mathfrak{D}-1]$-***sphere*** of radius $R \geqslant 0$ is the set of tuples

$$*: \qquad (x_1, x_2, \ldots, x_{\mathfrak{D}}) \in \mathbb{R}^{\times \mathfrak{D}}$$

with ${x_1}^2 + \cdots + {x_{\mathfrak{D}}}^2 = R^2$. In contrast, the $\mathfrak{D}$-***ball***, for $\mathfrak{D} \in \mathbb{N}$, comprises those $(*)$ with ${x_1}^2 + \cdots + {x_{\mathfrak{D}}}^2 \leqslant R^2$. So the $[\mathfrak{D}-1]$-sphere is the surface of the $\mathfrak{D}$-ball.

Examples. The 2-sphere is the usual sphere; the 1-sphere is the *circle*; the 0-sphere comprises the two points $\pm R$, which are the endpoints of the 1-*ball* $[-R, R]$, which is an *interval*.

The 2-ball is the *disk*, and the 3-ball is the usual (solid) ball, e.g a bowling ball. (The 0-ball is the one point of $\mathbb{R}^{\times 0}$; the empty tuple.) □

Prolegomena. Define constants $\beta_{\mathfrak{D}}, \sigma_{\mathfrak{D}}$, for $\mathfrak{D} = 0, 1, 2, \ldots$, from the ball of radius R:

$$\beta_{\mathfrak{D}} \cdot R^{\mathfrak{D}} := \mathfrak{D}\text{-volume of the } \mathfrak{D}\text{-ball;}$$
$$\sigma_{\mathfrak{D}} \cdot R^{\mathfrak{D}} := \mathfrak{D}\text{-SurfaceArea of } [\mathfrak{D}+1]\text{-ball}$$

For a ball, surface area is the growth rate of volume and so $\sigma_{\mathfrak{D}-1} R^{\mathfrak{D}-1}$ equals

$$\lim_{\Delta R \searrow 0} \frac{\beta_{\mathfrak{D}}[R + \Delta R]^{\mathfrak{D}} - \beta_{\mathfrak{D}} R^{\mathfrak{D}}}{\Delta R} \overset{\text{note}}{=\!=} \frac{\mathrm{d}}{\mathrm{d}R}(\beta_{\mathfrak{D}} \cdot R^{\mathfrak{D}}).$$

Consequently

$$1: \qquad \sigma_{\mathfrak{D}-1} = \mathfrak{D} \cdot \beta_{\mathfrak{D}}, \qquad \text{for } \mathfrak{D} \in \mathbb{Z}_+.$$

(Do we realize, for $\mathfrak{D} = 2, 1$, that we already knew (1); we just didn't know that we knew it?)

Let $J := \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$. We use the ***PCT*** ("polar coordinate trick") to show that $J = \sqrt{\pi}$. We integrate the cartesian-square of the integrand to conclude that

$$2: \quad \begin{aligned} J^2 &= \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[x^2]}\,\mathrm{d}x\right] \cdot \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[y^2]}\,\mathrm{d}y\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-[x^2+y^2]} \cdot \mathrm{d}(x, y) \\ &= \int_0^{+\infty} \mathrm{e}^{-R^2} \cdot \underbrace{2\pi R \cdot \mathrm{d}R}_{\substack{\text{Area of radius-}R\text{ annulus} \\ \text{of thickness } \mathrm{d}R.}} \end{aligned}$$

Hence $J^2 = \pi \cdot [-\mathrm{e}^{-R^2}]\Big|_{R=0}^{R=+\infty} = \pi$. Since J is the integral of a non-negative fnc, necessarily $J \geqslant 0$. Hence

$$3: \qquad \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \pi^{1/2}.$$

A Device used Twice is a Technique. For $\mathfrak{D}$ a posint, then, $\pi^{\mathfrak{D}/2}$ equals the product

$$\begin{aligned} [\pi^{\frac{1}{2}}]^{\mathfrak{D}} &= \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[x_1]^2}\,\mathrm{d}x_1\right] \cdot \ldots \cdot \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[x_{\mathfrak{D}}]^2}\,\mathrm{d}x_{\mathfrak{D}}\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-[{x_1}^2 + \cdots + {x_{\mathfrak{D}}}^2]} \cdot \mathrm{d}(x_1, \ldots, x_{\mathfrak{D}}). \end{aligned}$$

Converting this last integral into polar coordinates yields

$$2b: \qquad \pi^{\mathfrak{D}/2} = \int_0^{+\infty} \mathrm{e}^{-R^2} \cdot \underbrace{\sigma_{\mathfrak{D}-1} R^{\mathfrak{D}-1} \cdot \mathrm{d}R}_{\substack{\text{Infinitesimal "area" of} \\ \text{thickened } [\mathfrak{D}-1]\text{-sphere}}}.$$

Here $\sigma_{\mathfrak{D}-1} R^{\mathfrak{D}-1}\,\mathrm{d}R$ is the $[\mathfrak{D}-1]$-area of the spherical shell of radius R and thickness $\mathrm{d}R$. Let's now explore this last integral.

The Gamma Function

For each $\mathfrak{D} \in \mathbb{R}$, let $\boxed{I_{\mathfrak{D}} := \int_0^{+\infty} \mathrm{e}^{-R^2} R^{\mathfrak{D}-1}\,\mathrm{d}R}$. For $\mathfrak{D} \leqslant 0$ this integral is $+\infty$; we henceforth will $\boxed{\text{only consider } \mathfrak{D} \geqslant 0.}$ Courtesy the substitutions $2K + 2 := \mathfrak{D}$ and $t := R^2$, we have that

$$I_{\mathfrak{D}} = \tfrac{1}{2} \int_0^{\infty} \mathrm{e}^{-R^2} R^{2K} 2R\,\mathrm{d}R = \tfrac{1}{2} \int_0^{\infty} \mathrm{e}^{-t} t^K\,\mathrm{d}t.$$

This RhS suggests defining a fnc which is traditionally called the Gamma fnc:

$$4: \quad \Gamma(K+1) := \int_0^{\infty} t^K \mathrm{e}^{-t}\,\mathrm{d}t, \qquad \text{for } K \in (-1, \infty).$$

Hence $I_n = \frac{1}{2} \cdot \Gamma(\frac{n}{2})$. Solving for $\sigma_{\mathfrak{D}-1}$ in (2b) thus yields

$$5: \qquad \sigma_{\mathfrak{D}-1} = \frac{\pi^{\mathfrak{D}/2}}{I_n} = 2 \cdot \frac{\pi^{\mathfrak{D}/2}}{\Gamma(n/2)}.$$

Computing $\Gamma()$. Evidently $t^K[-\mathsf{e}^{-t}]\Big|_{t=0}^{t=\infty}$ is zero, once $K>0$. So integrating RhS(4) by parts yields: *For all real $K>0$,*

$$\Gamma(K+1) \;=\; 0 - \int_0^\infty [-\mathsf{e}^{-t}]\cdot Kt^K\,\mathrm{d}t \;=\; K\cdot\Gamma(K)\,.$$

Together with (5), this yields $\sigma_{\mathfrak{D}+1} = \sigma_{\mathfrak{D}-1}\cdot\frac{2\boldsymbol{\pi}}{\mathfrak{D}}$. Combining this with (1) gives

$$\textbf{6:}\qquad \begin{aligned}\sigma_{\mathfrak{D}} &= \sigma_{\mathfrak{D}-2}\cdot\tfrac{2}{\mathfrak{D}-1}\cdot\boldsymbol{\pi}\,,\\ \beta_{\mathfrak{D}} &= \beta_{\mathfrak{D}-2}\cdot\tfrac{2}{\mathfrak{D}}\cdot\boldsymbol{\pi}\,.\end{aligned}$$

for each $\mathfrak{D}\in[2\,..\,\infty)$.

SurArea and Vol of the $\mathfrak{D}$-ball

We know that $\sigma_0 = 2$ [an interval has 2 endpts] and that $\sigma_1 = 2\boldsymbol{\pi}$. So (6) allows us to fill-in the $\sigma_{\mathfrak{D}}$ column below. Then (1) allows us to fill-in the $\beta_{\mathfrak{D}}$ column.

$\mathfrak{D}$	$\beta_{\mathfrak{D}}\cdot R^{\mathfrak{D}}$	$\sigma_{\mathfrak{D}}\cdot R^{\mathfrak{D}}$
0	1	2
1	$2\cdot R$	$2\boldsymbol{\pi}\cdot R$
2	$\boldsymbol{\pi}\cdot R^2$	$4\boldsymbol{\pi}\cdot R^2$
3	$\frac{4}{3}\boldsymbol{\pi}\cdot R^3$	$2\boldsymbol{\pi}^2\cdot R^3$
4	$\frac{1}{2}\boldsymbol{\pi}^2\cdot R^4$	$\frac{8}{3}\boldsymbol{\pi}^2\cdot R^4$
5	$\frac{8}{15}\boldsymbol{\pi}^2\cdot R^5$	$\boldsymbol{\pi}^3\cdot R^5$
6	$\frac{1}{6}\boldsymbol{\pi}^3\cdot R^6$	$\frac{16}{15}\boldsymbol{\pi}^3\cdot R^6$
7	$\frac{16}{105}\boldsymbol{\pi}^3\cdot R^7$	$\frac{1}{3}\boldsymbol{\pi}^4\cdot R^7$
8	$\frac{1}{24}\boldsymbol{\pi}^4\cdot R^8$	$\frac{32}{105}\boldsymbol{\pi}^4\cdot R^8$
9	$\frac{32}{945}\boldsymbol{\pi}^4\cdot R^9$	$\frac{1}{12}\boldsymbol{\pi}^5\cdot R^9$
10	$\frac{1}{120}\boldsymbol{\pi}^5\cdot R^{10}$	$\frac{64}{945}\boldsymbol{\pi}^5\cdot R^{10}$
11	$\frac{64}{10395}\boldsymbol{\pi}^5\cdot R^{11}$	$\frac{1}{60}\boldsymbol{\pi}^6\cdot R^{11}$
12	$\frac{1}{720}\boldsymbol{\pi}^6\cdot R^{12}$	$\frac{128}{10395}\boldsymbol{\pi}^6\cdot R^{12}$

We can get a non-recursive formula for $\sigma_{\mathfrak{D}}$ if we are willing to use factorial notation.

Since $\int_0^\infty \mathsf{e}^{-t}\,\mathrm{d}t = 1$, we can summarize the foregoing as

$$\textbf{7:}\qquad \begin{aligned}\Gamma(0) &= +\infty\,;\\ \Gamma(1/2) &= \sqrt{\boldsymbol{\pi}}\,;\\ \Gamma(1) &= 1\,;\\ \Gamma(x) &= [x-1]\cdot\Gamma(x-1)\,, \qquad \text{for } x\in(0,\infty).\end{aligned}$$

Centroids

Our goal here is to find the centroid of a hemi-ball and hemi-sphere. Measuring from the center of the radius-R ball/sphere of dimension $\mathfrak{D}$, define constants $\mathbf{B}_{\mathfrak{D}}$ and $\mathbf{S}_{\mathfrak{D}}$ by

$\mathbf{B}_{\mathfrak{D}}\cdot R$:= Dist-to-centroid of the $\mathfrak{D}$-hemi-ball;
$\mathbf{S}_{\mathfrak{D}}\cdot R$:= Dist-to-centroid of the $\mathfrak{D}$-hemi-sphere.

We will compute this using an integral in polar coords of an angle θ. For $K\neq -1$, note that

$$\textbf{8:}\qquad \int_0^{\boldsymbol{\pi}/2}\sin(\theta)^K\cos(\theta)\cdot\mathrm{d}\theta \;=\; \frac{1}{K+1}\,.$$

Henceforth use $\mathbf{c}$ and $\mathbf{s}$ for $\cos(\theta)$ and $\sin(\theta)$. For the $[\mathfrak{D}+1]$-hemi-ball

$$\mathrm{Vol} \;=\; \int_0^{\boldsymbol{\pi}/2} \underbrace{\beta_{\mathfrak{D}}\cdot[R\mathbf{s}]^{\mathfrak{D}}}_{\text{Area of slice.}}\cdot\underbrace{\mathbf{s}\cdot R\,\mathrm{d}\theta}_{\text{Thickness of slice.}}\,.$$

We already know that the value of this integral is $\frac{1}{2}\beta_{\mathfrak{D}+1}\cdot R^{\mathfrak{D}+1}$. The *Total Torque Relative to Zero* is computed by the same integral, together with a lever arm.

$$TTRZ = \int_0^{\boldsymbol{\pi}/2}\underbrace{\beta_{\mathfrak{D}}\cdot[R\mathbf{s}]^{\mathfrak{D}}}_{\text{Area.}}\cdot\underbrace{\mathbf{s}\cdot R\,\mathrm{d}\theta}_{\text{Thickness.}}\cdot\underbrace{R\mathbf{c}}_{\text{Lever arm.}}\,.$$

Courtesy (8),

$$\textbf{9:}\qquad TTRZ \;=\; R^{\mathfrak{D}+2}\beta_{\mathfrak{D}}\cdot\frac{1}{\mathfrak{D}+2}\,.$$

The quotient gives the centroid.

$$\textbf{10:}\qquad \mathbf{B}_{\mathfrak{D}+1} \;=\; \frac{\mathrm{TTRZ}}{\mathrm{Vol}} \;=\; \frac{\beta_{\mathfrak{D}}}{\beta_{\mathfrak{D}+1}}\cdot\frac{2}{\mathfrak{D}+2}\,.$$

♦

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