

Volume of the \mathfrak{D} -ball and the Gamma function

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage <http://squash.1gainesville.com/>
19 November, 2017 (at 12:53)

ABSTRACT: (See “Joe’s Circle Puzzle” for an application of this, in PROBLEMS NB)

We are born knowing that... For $\mathfrak{D} \in \mathbb{Z}_+$, the $[\mathfrak{D}-1]$ -sphere of radius $r \geq 0$ is the set of tuples

$$\forall: (x_1, x_2, \dots, x_{\mathfrak{D}}) \in \mathbb{R}^{\times \mathfrak{D}}$$

with $x_1^2 + \dots + x_{\mathfrak{D}}^2 = r^2$. In contrast, the [closed] \mathfrak{D} -ball, for $\mathfrak{D} \in \mathbb{N}$, comprises those (\forall) with $x_1^2 + \dots + x_{\mathfrak{D}}^2 \leq r^2$. So the $[\mathfrak{D}-1]$ -sphere is the surface of the \mathfrak{D} -ball.

Examples. The 2-sphere is the usual sphere; the 1-sphere is the *circle*; the 0-sphere comprises the two points $\pm r$, which are the endpoints of the 1-ball $[-r, r]$, which is an *interval*.

The 2-ball is the *disk*, and the 3-ball is the usual (solid) ball, e.g a bowling ball. (The 0-ball is the one point of $\mathbb{R}^{\times 0}$; the empty tuple.) \square

Prolegomena. Define constants $\beta_{\mathfrak{D}}, \sigma_{\mathfrak{D}}$, for $\mathfrak{D} = 0, 1, 2, \dots$, from the ball of radius r :

$$\begin{aligned} \beta_{\mathfrak{D}} \cdot r^{\mathfrak{D}} &:= \mathfrak{D}\text{-volume of the } \mathfrak{D}\text{-ball;} \\ \sigma_{\mathfrak{D}} \cdot r^{\mathfrak{D}} &:= \mathfrak{D}\text{-SurfaceArea of } [\mathfrak{D}+1]\text{-ball} \end{aligned}$$

For a ball, surface area is the growth rate of volume and so $\sigma_{\mathfrak{D}-1} r^{\mathfrak{D}-1}$ equals

$$\lim_{\Delta r \searrow 0} \frac{\beta_{\mathfrak{D}}[r + \Delta r]^{\mathfrak{D}} - \beta_{\mathfrak{D}} r^{\mathfrak{D}}}{\Delta r} \stackrel{\text{note}}{=} \frac{d}{dr}(\beta_{\mathfrak{D}} \cdot r^{\mathfrak{D}}).$$

Consequently

$$1: \quad \sigma_{\mathfrak{D}-1} = \mathfrak{D} \cdot \beta_{\mathfrak{D}}, \quad \text{for } \mathfrak{D} \in \mathbb{Z}_+.$$

(Do we realize, for $\mathfrak{D} = 2, 1$, that we already knew (1); we just didn’t know that we knew it?)

Let $J := \int_{-\infty}^{+\infty} e^{-x^2} dx$. We use the *PCT* (“polar coordinate trick”) to show that $J = \sqrt{\pi}$. We integrate the cartesian-square of the integrand to conclude that

$$\begin{aligned} J^2 &= \left[\int_{-\infty}^{+\infty} e^{-[x^2]} dx \right] \cdot \left[\int_{-\infty}^{+\infty} e^{-[y^2]} dy \right] \\ 2.1: \quad &= \int_{-\infty}^{+\infty} e^{-[x^2+y^2]} \cdot d(x, y) \\ &= \int_0^{+\infty} e^{-r^2} \cdot \underbrace{2\pi r \cdot dr}_{\substack{\text{Area of radius-}r \text{ annulus} \\ \text{of thickness } dr.}} \end{aligned}$$

Hence $J^2 = \pi \cdot [-e^{-r^2}]_{r=0}^{r=+\infty} = \pi$. Since J is the integral of a non-negative fnc, nec. $J \geq 0$. Thus

$$2.2: \quad \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$$

A Device used Twice is a Technique.

For \mathfrak{D} a posint, then, $\pi^{\mathfrak{D}/2}$ equals the product

$$\begin{aligned} [\pi^{\frac{1}{2}}]^{\mathfrak{D}} &= \left[\int_{-\infty}^{+\infty} e^{-[x_1]^2} dx_1 \right] \cdot \dots \cdot \left[\int_{-\infty}^{+\infty} e^{-[x_{\mathfrak{D}}]^2} dx_{\mathfrak{D}} \right] \\ &= \int_{-\infty}^{+\infty} e^{-[x_1^2 + \dots + x_{\mathfrak{D}}^2]} \cdot d(x_1, \dots, x_{\mathfrak{D}}). \end{aligned}$$

Converting this last integral into polar coordinates yields

$$2.1b: \quad \pi^{\mathfrak{D}/2} = \int_0^{+\infty} e^{-r^2} \cdot \underbrace{\sigma_{\mathfrak{D}-1} r^{\mathfrak{D}-1}}_{\substack{\text{Infinitesimal } \mathfrak{D}\text{-volume of} \\ \text{thickened } [\mathfrak{D}-1]\text{-sphere.}}} \cdot dr.$$

Here, $\sigma_{\mathfrak{D}-1} r^{\mathfrak{D}-1}$ is the $[\mathfrak{D}-1]$ -area of the spherical shell of radius r . Multiplying by thickness dr gives $\sigma_{\mathfrak{D}-1} r^{\mathfrak{D}-1} dr$ as the infinitesimal \mathfrak{D} -volume of the thickened sphere.

Let’s explore this last integral.

The Gamma Function

For each $\mathfrak{D} \in \mathbb{R}$, let $I_{\mathfrak{D}} := \int_0^{+\infty} e^{-r^2} r^{\mathfrak{D}-1} dr$. For $\mathfrak{D} \leq 0$ this integral is $+\infty$; we henceforth will only consider $\mathfrak{D} \geq 0$. Courtesy the substitutions $2K + 2 := \mathfrak{D}$ and $t := r^2$, we have that

$$I_{\mathfrak{D}} = \frac{1}{2} \int_0^{\infty} \underbrace{e^{-r^2}}_{e^{-t}} \underbrace{r^{2K}}_{t^K} \underbrace{2r dr}_{dt} = \frac{1}{2} \int_0^{\infty} e^{-t} t^K dt.$$

This RhS suggests defining a fnc which is traditionally called the *Gamma function*:

$$*: \quad \Gamma(K + 1) := \int_0^{\infty} t^K e^{-t} dt, \quad \text{for } K \in \mathbb{C} \text{ with } \text{Re}(K) > -1.$$

So $I_{2K+2} = \frac{1}{2} \cdot \Gamma(K + 1)$. Hence $I_{\mathfrak{D}} = \frac{1}{2} \cdot \Gamma(\frac{\mathfrak{D}}{2})$. Solving for $\sigma_{\mathfrak{D}-1}$ in (2.1b) thus yields

$$1b: \quad \sigma_{\mathfrak{D}-1} = \frac{\pi^{\mathfrak{D}/2}}{I_{\mathfrak{D}}} = 2 \cdot \frac{\pi^{\mathfrak{D}/2}}{\Gamma(\mathfrak{D}/2)}.$$

Computing $\Gamma(\cdot)$. Easily, $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$.
 When $\text{Re}(K) > 0$, then $\lim_{t \searrow 0} [t^K e^{-t}]$ is zero. Thus

$$t^K [-e^{-t}] \Big|_{t=0}^{t=\infty} = 0.$$

Integrating $\text{RHS}(\ast)$ by parts produces $\Gamma(K + 1) = 0 - \int_0^\infty [-e^{-t}] \cdot K t^{K-1} dt$. Thus

For all $K \in \mathbb{C}$
 with $\text{Re}(K) > 0$: $\Gamma(K + 1) = K \cdot \Gamma(K)$.

Together with (1b), this yields $\sigma_{\mathfrak{D}+1} = \sigma_{\mathfrak{D}-1} \cdot \frac{2\pi}{\mathfrak{D}}$.
 Combining this with (1) gives

1c:
$$\begin{aligned} \sigma_{\mathfrak{D}} &= \sigma_{\mathfrak{D}-2} \cdot \frac{2}{\mathfrak{D}-1} \cdot \pi, \\ \beta_{\mathfrak{D}} &= \beta_{\mathfrak{D}-2} \cdot \frac{2}{\mathfrak{D}} \cdot \pi. \end{aligned}$$

for each $\mathfrak{D} \in [2.. \infty)$.

SurArea and Vol of the \mathfrak{D} -ball

We know that $\sigma_0 = 2$ [an interval has 2 endpts] and that $\sigma_1 = 2\pi$. So (1c) allows us to fill-in the $\sigma_{\mathfrak{D}}$ column below. Then (1) allows us to fill-in the $\beta_{\mathfrak{D}}$ column.

\mathfrak{D}	$\beta_{\mathfrak{D}} \cdot r^{\mathfrak{D}}$	$\sigma_{\mathfrak{D}} \cdot r^{\mathfrak{D}}$
0	1	2
1	$2 \cdot r$	$2\pi \cdot r$
2	$\pi \cdot r^2$	$4\pi \cdot r^2$
3	$\frac{4}{3} \pi \cdot r^3$	$2\pi^2 \cdot r^3$
4	$\frac{1}{2} \pi^2 \cdot r^4$	$\frac{8}{3} \pi^2 \cdot r^4$
5	$\frac{8}{15} \pi^2 \cdot r^5$	$\pi^3 \cdot r^5$
6	$\frac{1}{6} \pi^3 \cdot r^6$	$\frac{16}{15} \pi^3 \cdot r^6$
7	$\frac{16}{105} \pi^3 \cdot r^7$	$\frac{1}{3} \pi^4 \cdot r^7$
8	$\frac{1}{24} \pi^4 \cdot r^8$	$\frac{32}{105} \pi^4 \cdot r^8$
9	$\frac{32}{945} \pi^4 \cdot r^9$	$\frac{1}{12} \pi^5 \cdot r^9$
10	$\frac{1}{120} \pi^5 \cdot r^{10}$	$\frac{64}{945} \pi^5 \cdot r^{10}$
11	$\frac{64}{10395} \pi^5 \cdot r^{11}$	$\frac{1}{60} \pi^6 \cdot r^{11}$
12	$\frac{1}{720} \pi^6 \cdot r^{12}$	$\frac{128}{10395} \pi^6 \cdot r^{12}$

We can get a non-recursive formula for $\sigma_{\mathfrak{D}}$ using factorial notation. [\[Exercise.\]](#)

$\Gamma(\cdot)$ summary. Function

2.2b: $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \text{for } z \in \mathbb{C} \text{ with } \text{Re}(z) > 0,$

has values

$$\Gamma(0) = \infty;$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi};$$

2.2c: $\Gamma(1) = 1;$

$$\Gamma(n) = [n-1]!, \quad \text{for } n \in \mathbb{Z}_+. \text{ More generally,}$$

$$\Gamma(z) = [z-1] \cdot \Gamma(z-1), \quad \text{for } \text{Re}(z) > 1.$$

Centroids

Our goal here is to find the centroid of a hemi-ball and hemi-sphere. Measuring from the center of the radius- r ball/sphere of dimension \mathfrak{D} , define constants $\mathbf{B}_{\mathfrak{D}}$ and $\mathbf{S}_{\mathfrak{D}}$ by

$\mathbf{B}_{\mathfrak{D}} \cdot r :=$ Dist-to-centroid of the \mathfrak{D} -hemi-ball;

$\mathbf{S}_{\mathfrak{D}} \cdot r :=$ Dist-to-centroid of the \mathfrak{D} -hemi-sphere.

We will compute this using an integral in polar coords of an angle θ . For $K \neq -1$, note that

3:
$$\int_0^{\pi/2} \sin(\theta)^K \cos(\theta) \cdot d\theta = \frac{1}{K+1}.$$

Henceforth use \mathbf{c} and \mathbf{s} for $\cos(\theta)$ and $\sin(\theta)$. For the $[\mathfrak{D}+1]$ -hemi-ball

$$\text{Vol} = \int_0^{\pi/2} \underbrace{\beta_{\mathfrak{D}} \cdot [rs]^{\mathfrak{D}}}_{\text{Area of slice}} \cdot \underbrace{\mathbf{s} \cdot r d\theta}_{\text{Thickness of slice}}.$$

We already know that the value of this integral is $\frac{1}{2} \beta_{\mathfrak{D}+1} \cdot r^{\mathfrak{D}+1}$. The *Total Torque Relative to Zero* is computed by the same integral, together with a lever arm.

$$\text{TTRZ} = \int_0^{\pi/2} \underbrace{\beta_{\mathfrak{D}} \cdot [rs]^{\mathfrak{D}}}_{\text{Area}} \cdot \underbrace{\mathbf{s} \cdot r d\theta}_{\text{Thickness}} \cdot \underbrace{rc}_{\text{Lever arm}}.$$

Courtesy (3),

4:
$$\text{TTRZ} = r^{\mathfrak{D}+2} \beta_{\mathfrak{D}} \cdot \frac{1}{\mathfrak{D}+2}.$$

The quotient gives the centroid.

$$5: \quad \mathbf{B}_{\mathfrak{D}+1} = \frac{\text{TTRZ}}{\text{Vol}} = \frac{\beta_{\mathfrak{D}}}{\beta_{\mathfrak{D}+1}} \cdot \frac{2}{\mathfrak{D} + 2}.$$



Filename: Problems/Geometry/vol.nball.latex
As of: Monday 20Feb2006. Typeset: 19Nov2017 at 12:53.