Vector Fields

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Prolegomenon. We have an innerproduct space $\mathbf{V}$; use $\mathbf{V} := \mathbb{R}^2$ for the time being. A subset $\Omega \subset \mathbf{V}$ is a ***domain*** if Ω is non-void, open and connected. (The "connected" isn't really necessary.)

Henceforth J denotes a closed interval $[a, b]$ in $\mathbb{R}$. A ***path*** Γ (in Ω) can be described by a *piecewise-* $\mathbf{C}^1$ parameterization $\mathbf{r}: J \to \Omega$. Path Γ starts at point $\mathrm{Start}(\Gamma) := \mathbf{r}(a)$ and ends at $\mathrm{Stop}(\Gamma) := \mathbf{r}(b)$ (So, technically, a path can be viewed as an equivalence class of parameterizations.)

Use $-\Gamma$ for Γ traversed backwards. Each parameterization $\mathbf{r}: J \to \Omega$ of Γ yields a parameterization $\mathbf{q}: J \to \Omega$ of $-\Gamma$ by defining

$$\mathbf{q}(t) := \mathbf{r}(b+a-t).$$

A "***vectorfield*** $\mathbf{F}$ on Ω" is a mapping $\mathbf{F}: \Omega \to \mathbf{V}$; usually we ask that the v.f be at least cts (continuous).

The ***path integral*** of a v.f over a path is

$$\int_\Gamma \langle \mathbf{F}, \mathrm{d}\mathbf{r} \rangle := \int_J \langle \mathbf{F}(\mathbf{r}(t)), \mathbf{r}'(t) \rangle \, \mathrm{d}t.$$

A field is ***conservative*** if, for each *closed loop* Γ, the integral $\int_\Gamma \langle \mathbf{F}, \mathrm{d}\mathbf{r} \rangle$ is zero.

A field $\mathbf{F}$ is a ***gradient v.f*** if there exists a $\mathbf{C}^1$-function $h: \Omega \to \mathbb{R}$ so that $\nabla h = \mathbf{F}$. This h is called a ***potential-fnc*** of $\mathbf{F}$. (Evidently $h+17$ is another potential-fnc for $\mathbf{F}$.)

1: Fund. Thm of Path Integrals (FTPI). *On Ω,*

$$\int_\Gamma \langle \nabla h, \mathrm{d}\mathbf{r} \rangle = h(\mathrm{Stop}(\Gamma)) - h(\mathrm{Start}(\Gamma))$$

for each potential-function h and each path Γ. ◊

Write a vectorfield as $\mathbf{F} = \alpha\hat{\imath} + \beta\hat{\jmath}$, where $\alpha, \beta: \Omega \to \mathbf{V}$ are the ***component fncs*** of $\mathbf{F}$. Suppose this $\mathbf{F} = \nabla h$. Then $h_x = \alpha$ and $h_y = \beta$. By equality (Clairaut's thm) of the mixed 2^{nd}-partials of h, automatically $\beta_x = \alpha_y$. The difference

$$\text{2:} \qquad \text{k-curl}_{\mathbf{F}}() := \beta_x() - \alpha_y()$$

is a map $\Omega \to \mathbb{R}$ which will be useful in the sequel. We read 'k-curl$_{\mathbf{F}}(P)$' as *"the **k-curl** of $\mathbf{F}$ at (point) P"*.

For a gradient v.f, the k-curl[♡1] is identically zero; such a field is said to be ***irrotational***.

Example 1. For a vector $\mathbf{w} \in \mathbb{R}^2$, let $\mathrm{Turn}(\mathbf{w})$ be the vector which is $\mathbf{w}$ rotated CCW (counterclockwise) by a right-angle.

Let σ be a **"**strength fnc**"** on the posreals. Define a corresponding v.f $\mathbf{F}_\sigma$ by

$$\mathbf{F}_\sigma(\mathbf{w}) := \sigma(\|\mathbf{w}\|) \cdot \mathrm{Turn}(\mathbf{w}) / \|\mathbf{w}\|.$$

Letting C_ρ be the radius-ρ circle oriented CCW, we have that

$$\text{3:} \qquad \int_{C_\rho} \langle \mathbf{F}_\sigma, \mathrm{d}\mathbf{r} \rangle = \sigma(\rho) \cdot 2\pi\rho.$$

So certainly $\mathbf{F}_\sigma$ is <u>not</u> conservative. □

Example 2. Take a σ so that integral (3) is independent of ρ; say $\sigma(\rho) := 1/\rho$. Let $\mathbf{G} := \mathbf{F}_\sigma$ be the corresponding vectorfield; it is defined on the *punctured plane*, which we will write as Ω°. It can be shown that

$$\int_\Gamma \langle \mathbf{G}, \mathrm{d}\mathbf{r} \rangle = 0$$

for each closed-loop $\Gamma \subset \Omega^\circ$ which *does not go around the origin.*[♡2] (Proof-idea: Approximate Γ by a finite sequence of circle-arcs and radial segments.) In x, y-coordinates our $\mathbf{G}$ equals $\alpha\hat{\imath} + \beta\hat{\jmath}$, where

$$\text{4:} \qquad \begin{aligned} \alpha &= -y/[x^2+y^2] \quad \text{and} \\ \beta &= x/[x^2+y^2]. \end{aligned}$$

Exercise: Use (4) and (2) to verify that $\mathbf{G}$ is irrotational.

To what extent is there a real-valued fnc h so that $\nabla h = \mathbf{G}$? Expressed in polar-coords, suppose that $Q := (\rho, \theta)$ is a particular point in Ω°. The map $\varphi \mapsto (\rho, \varphi)$, as angle φ strolls from 0 to θ, parameterizes an oriented arc of circle, call it Γ_Q, going from point $(\rho, 0)$ to Q. Hence

$$\int_{\Gamma_Q} \langle \mathbf{F}_\sigma, \mathrm{d}\mathbf{r} \rangle = \theta,$$

[♡1] Difference (2) is the $\widehat{\mathbf{k}}$-component of the curl of a 3-dim vectorfield, whence the name k-curl.

[♡2] Indeed, more generally, the integral is zero for each closed-loop Γ with *winding number* zero.

and does not depend on ρ. So the definition

5: $$h(\rho, \theta) \;:=\; \theta$$

would give a potential-fnc h for the $\mathbf{G}$ vectorfield —*if* we had a consistent way to assign real numbers to geometric angles on a circle. The problem is that *distinct* reals, e.g θ and $\theta+2\boldsymbol{\pi}$, name the same geometric angle.

To finesse this difficulty, agree to slit the plane along a ray from the origin. On this slit-plane our (restricted) $\mathbf{G}$ *is* a gradient v.f. We can rewrite fnc (5) in cartesian coords as

5′: $$h(x, y) \;:=\; \arctan\left(\frac{y}{x}\right).$$

Comparing to (4), one can verify[♡3] that $h_x = \alpha$ and $h_y = \beta$, at those points where (**??**′) is well-defined. □

Defn. A domain $\Omega \subset \mathbb{R}^N$ is ***simply-connected*** if each closed-loop $\Gamma \subset \Omega$ can be contracted (always staying within Ω) to a point. For example, a sphere or punctured sphere is simply-connected. However a *doubly*-punctured sphere is not. Neither a torus nor a solid-torus is simply-connected. □

6: Theorem. *Suppose that* $\mathbf{F}\colon\Omega\to\mathbb{R}$ *is a continuous vectorfield in* $\mathbb{R}^N$. *Then*

$$[\mathbf{F} \text{ a gradient v.f.}] \iff [\mathbf{F} \text{ conservative}]. \quad \Diamond$$

The following thm generalizes to higher dimensions, but we'll state in in dimension 2.

7: Theorem. *Suppose that* $\mathbf{F}$ *is a* $\mathbf{C}^1$*-vectorfield on* $\Omega \subset \mathbb{R}^2$. *Then*

$$[\mathbf{F} \text{ a gradient v.f}] \implies [\mathbf{F} \text{ is irrotational}].$$

If Ω *is simply-connected, then the converse holds.* ◊

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[♡3] That y/x is undefined at $x = 0$ is an artifact of using cartesian coordinates (and the arbitrary choice of domain for arctan). After all, the expression $\arctan(y/x)$ extends continuously over a slit plane.