Numbers & Polys MAS3300 | **Home-V** | Prof. JLF King *Touch*: 21Feb2017

Note. Permitted: Brain, SigmonNotes, calculator, computer, webpage; but no other human beings other than me. Please write each essay on separate sheets of paper, using complete grammatical English sentences. Use **every third line**.

Essays violate the Checklist at *Grade Peril*!

Write expressions unambiguously e.g, "$1/a + b$" should be bracketed either $[1/a] + b$ or $1/[a + b]$. (Be careful with **negative** signs!) Every "**if**" must be matched by a "**then**".

Use $\mathcal{P}(S)$ to mean "the powerset of S".

Exam is due no later than **10:00*AM*, Friday, 28Apr2006**, slid under LIT402. Please email me afterwards.

V1: Suppose that α is a root of $\mathbb{Q}$-irreducible poly

$$f(x) := x^3 - x^2 - 3x - 2 .$$

Fact: Each number $z \in \mathbb{Q}(\!(\alpha)\!)$ can be written in ***std. form***: as

$$\mathrm{SF}(z) = q_2\alpha^2 + q_1\alpha + q_0$$

where each $q_i \in \mathbb{Q}$. **a** Please fill-in

$$\mathrm{SF}(\alpha^4) = \_\_\_\_ \alpha^2 + \_\_\_\_ \alpha + \_\_\_\_ .$$

$$\mathrm{SF}(2\alpha^5 - \alpha^3) = \_\_\_\_ \alpha^2 + \_\_\_\_ \alpha + \_\_\_\_ .$$

$$\mathrm{SF}\left(\frac{1}{\alpha}\right) = \_\_\_\_ \alpha^2 + \_\_\_\_ \alpha + \_\_\_\_ .$$

b Let $h(x) := x^2 + 2x - 1$. This $h()$ is coprime to $f()$. Use ↯ to compute ratpolys $S(x)=$ ________ and $T(x)=$ ________, st. $S \cdot f + T \cdot h = 1$. (Verify this!) Use this information to write

$$\mathrm{SF}\left(\frac{1}{\alpha^2 + 2\alpha - 1}\right) = \_\_\_\_ \alpha^2 + \_\_\_\_ \alpha + \_\_\_\_ .$$

V2: Carefully write a formal proof that there exist *ir*rational positive reals w, z so that w^z is *rational*, as follows: Let

$$S := \sqrt{7},\ \ T := \sqrt{2} \quad \text{and} \quad P := S^T .$$

Argue that either S^T is such an example, or P^T is.

V3: Consider the Fibonacci numbers $(\mathsf{f}_n)_{n=-\infty}^{\infty}$ defined by $\mathsf{f}_0 := 0$, $\mathsf{f}_1 := 1$, and $\forall n \in \mathbb{Z} : \mathsf{f}_{n+1} = \mathsf{f}_n + \mathsf{f}_{n-1}$. Prove by induction that

$$*: \qquad \forall n \in \mathbb{Z}_+ : \quad [\mathsf{f}_n]^2 + [\mathsf{f}_{n-1}]^2 = \mathsf{f}_{2n-1} ,$$

or provide a CEX. Does $(*)$ hold for n negative?

V4: For any sets C, D, let "C^D" be the set of functions $f: D \to C$. (Think Domain→Codomain.). **a** For arbitrary sets A,P,Q, construct an *explicit* bijection (with *proof*!)

$$\mathbf{H}: A^{P\times Q} \hookrightarrow\!\!\!\!\rightarrow [A^P]^Q .$$

That is, given an arb. fnc $f \in A^{P\times Q}$, your construction/defn produces a *specific, explicit* fnc $\mathbf{H}(f)$ in $[A^P]^Q$.

[*Suggestion:* To get the idea, first consider specific sets, e.g $A := \{0,1\}$, $P := \{C,L,M\}$ and $Q := \{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$.] Once you get the idea, write your argument for general sets. Remember that that any of these sets could be uncountable (so you can NOT list its elts), or empty.

b (Below, we need any two 2-elts sets. I chose $\{0,1\}$ and $\{6,7\}$.) Use "2^A" to abbrev. $\{0,1\}^A$. You are given bijections

$$\mathcal{F}: \mathbb{N}\times\{6,7\} \hookrightarrow\!\!\!\!\rightarrow \mathbb{N} \quad \text{and} \quad \mathcal{G}: 2^{\mathbb{N}} \hookrightarrow\!\!\!\!\rightarrow \mathbb{R} .$$

In terms of $\mathcal{F}$, $\mathcal{P} := \mathcal{F}^{-1}$, $\mathcal{G}$, $\mathcal{Q} := \mathcal{G}^{-1}$ and *your construction* (*above*), define these maps **explicitly**:

1: $\beta: \mathbb{R}^2 \hookrightarrow\!\!\!\!\rightarrow 2^{\mathbb{N}}\times 2^{\mathbb{N}}$.

2: $\gamma: 2^{\mathbb{N}}\times 2^{\mathbb{N}} \hookrightarrow\!\!\!\!\rightarrow 2^{\mathbb{N}\times\{6,7\}}$.

3: $\delta: 2^{\mathbb{N}\times\{6,7\}} \hookrightarrow\!\!\!\!\rightarrow 2^{\mathbb{N}}$.

Use all this to give an *explicit* bijection

4: $\varepsilon: \mathbb{R}^2 \hookrightarrow\!\!\!\!\rightarrow \mathbb{R}$.

[*Hint:* The γ fnc could use your construction.]

V1:	___ ___ ___	110pts
V2:	___ ___	80pts
V3:	___ ___	80pts
V4:	___ ___ ___	125pts
Total:	___ ___ ___	395pts

Print name ________________ Ord:

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: ________________