Staple!

Ord: ___ ___

Plex
MAA4402 8436

Class-V

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Wedn, 06Oct2021

Notation. All sets are subsets of $\mathbb{C}$. For sets B and E, the difference set is

$$B \smallsetminus E \quad := \quad \left\{ x \in B \;\middle|\; x \notin E \right\}.$$

The complement of E is $E^c := \mathbb{C} \smallsetminus E$.

For short-answer: Write **DNE** <u>if</u> the object does not exist or the operation cannot be performed. NB: **DNE** $\neq \{\} \neq 0$.

V1: Short answer. Show no work. $\mathbb{C}$-plane

a) 10 10 Number $\left[\mathbf{i} + \sqrt{3}\right]^{70} = x + \mathbf{i}y$, for real numbers $x=$ .... and $y=$ ....

[*Multiplying complexes multiplies their moduli (absolute-values), and adds their angles.*]

Powers: For $\alpha := \mathbf{i} + \sqrt{3}$, note $|\alpha| = \sqrt{1^2+3} = 2$. So $u := \frac{\alpha}{2} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i}$ has length 1, with angle

$$\operatorname{Arg}(u) = \boldsymbol{\pi}/6 = 30^\circ. \quad \textit{i.e,}$$
$$u = \exp(\tfrac{\boldsymbol{\pi}}{6}\mathbf{i}). \quad \textit{Thus} \quad u^{12} = 1.$$

So $u^{70} = u^{72} \cdot u^{-2} = u^{-2} = \exp(-\frac{\boldsymbol{\pi}}{3}\mathbf{i}) = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}.$

Hence

$$\alpha^{70} = 2^{70}u^{70} = 2^{69} \cdot [1 - \sqrt{3}\mathbf{i}].$$

Thus $x = 2^{69}$ and $y = -2^{69} \cdot \sqrt{3}$. *Nifty!*

ADDENDUM: Since $\alpha = 2\,\mathrm{e}^{\frac{1}{6}\boldsymbol{\pi}\mathbf{i}}$. our α^{70} is 2^{70} times $\mathrm{e}^{\frac{70}{6}\boldsymbol{\pi}\mathbf{i}}$, i.e, times

$$\cos(\tfrac{70}{6}\boldsymbol{\pi}) + \mathbf{i}\sin(\tfrac{70}{6}\boldsymbol{\pi}) \overset{\text{note}}{=\!=} \cos(\tfrac{-\boldsymbol{\pi}}{3}) + \mathbf{i}\sin(\tfrac{-\boldsymbol{\pi}}{3})$$
$$= \frac{1}{2} - \mathbf{i}\frac{\sqrt{3}}{2}.$$

15 10 b Fnc $u(x,y) := \cos(y \cdot x) - 7x$ maps $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Its Laplacian is $[\Delta(u)](x,y)=$ $-\cos(y \cdot x) \cdot [y^2 + x^2]$.

There exists function $v{:}\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $f(x+\mathbf{i}y) := u(x,y) + \mathbf{i}v(x,y)$ is holomorphic. T F

Lapl Soln: Let $\mathcal{C} := \cos(y \cdot x)$ and $\mathcal{S} := \sin(y \cdot x)$.

Note $\Delta(u) = \Delta(\mathcal{C}) = \mathcal{C}_{xx} + \mathcal{C}_{yy}$, since the $7x$ is killed-off by 2^{nd}-differentiation.

The Chain rule gives $\mathcal{C}_x = -\mathcal{S}\cdot y$. Hence $\mathcal{C}_{xx} = -\mathcal{C}\cdot y^2$. As $\mathcal{C}$ is symmetric in x and y, we get for free that $\mathcal{C}_{yy} = -\mathcal{C}\cdot x^2$. Adding, $\Delta(u) = \Delta(\mathcal{C}) = -\cos(y\cdot x)\cdot[y^2 + x^2]$.

As $\Delta(u) \not\equiv \mathbf{0}$, our u is not harmonic. Hence, by our thm from class, this u is ***not*** the real part of a holomorphic fnc.

15 10 C Write $\cos(-3\mathbf{i})$, which is real, ITOf exp() and *finite* add/sub/mul/div: $\cos(-3\mathbf{i})$= ....

And $\cos(-3\mathbf{i})$ lies in circle the correct interval

$(-\infty, \frac{-1}{5}]$ $(\frac{-1}{5}, \frac{1}{5}]$ $(\frac{1}{5}, 2]$ $(2, 5]$ $(5, 15]$ $(15, 45]$ $(45, \infty)$

Soln: Recall that $\cos(z) = [\mathsf{e}^{\mathbf{i}z} + \mathsf{e}^{-\mathbf{i}z}]/2$, for all $z \in \mathbb{C}$. Hence $\cos(-3\mathbf{i})$ equals

$$\frac{1}{2}\mathsf{e}^{-3} \quad + \quad \frac{1}{2}\mathsf{e}^{3}.$$

The $\mathsf{e}^{-3}/2$ term is negligible, here. As $\frac{5}{2} < \mathsf{e} < 3$, so $\frac{125}{8} < \mathsf{e}^3 < 27$. Thus $5 < \frac{125}{16} < \frac{1}{2}\mathsf{e}^3 < \frac{27}{2} < 15$.

25 d Compute the real $\alpha=$ ⌞. . . . .⌟ such that

$$*: \qquad 3^\alpha \cdot \sum_{k=0}^{1801} \binom{1801}{k} 2^k \quad = \quad \sum_{j=0}^{892} \binom{892}{j} 8^j \,.$$

[*Hint:* The Binomial Theorem]

Binom Soln: LhS(∗) equals $3^\alpha \cdot [2+1]^{1801} \;=\; 3^{\alpha+1801}$. RhS(∗) equals $[8+1]^{892} \overset{\text{note}}{=\!=} 3^{[2\cdot 892]}$. Consequently,

$$\alpha \;=\; [2\cdot 892] - 1801 \;=\; -17\,.$$

[The Binomial thm is on P.30 of *PlexNotes*.]

1 9 10 e The number of permutations of "PREPPER", as a multinomial coefficient, is $\binom{7}{3,2,2} = \binom{7}{3}\binom{4}{2}\binom{2}{2} = 35 \cdot 6 \cdot 1 \overset{\text{numeral}}{=\!=\!=} 210$.

Soln: Pockets are labeled 'P', 'R', 'E'. Tokens placed in the pockets are letter-positions 1,2,3,4,5,6,7. Hence $\binom{7}{3,2,2}$ is the number of permutations of "PREPPER".

V2: Short answer. Metric space stuff

25 f *The empty-set is connected:* (T) F

Punctured ball $\mathrm{PBal}_2(3\boldsymbol{i})$ *is connected:* (T) F

$\mathrm{Sph}_2(5\boldsymbol{i}) \cap \mathrm{Sph}_2(\boldsymbol{i})$ *is connected:* (T) F

$\mathrm{Sph}_2(4\boldsymbol{i}) \cup \mathrm{Sph}_2(-\boldsymbol{i})$ *is connected:* T (F)

$\mathrm{Sph}_2(5\boldsymbol{i}) \cup \mathrm{CldBal}_2(\boldsymbol{i})$ *is closed:* (T) F

Soln: In $\mathbb{C}$, a punctured-ball is path-connected, so is certainly connected.

Since $|5\mathbf{i} - \mathbf{i}| = 2{\cdot}2$, intersection $\mathrm{Sph}_2(5\mathbf{i}) \cap \mathrm{Sph}_2(\mathbf{i})$ is singleton $\{3\mathbf{i}\}$; connected. But $|4\mathbf{i} - {-\mathbf{i}}| > 2{\cdot}2$, so *union* $\mathrm{Sph}_2(4\mathbf{i}) \cup \mathrm{Sph}_2(-\mathbf{i})$ comprises two *not*-touching circles, hence is disconnected.

Recall that the union of *finitely many* closed sets is closed. Hence $\mathrm{Sph}_2(5\mathbf{i}) \cup \mathrm{CldBal}_2(\mathbf{i})$ is closed.

g 25 All these sets are non-empty: Sets U and V are open. Sets K, E and E_n are closed. Sets A and B are each path-connected.

$\exists q \in [A \cap B]$; *so* $A \cap B$ *is path-connected:*	AT	AF	(Nei)
Union $\bigcup_{n=1}^{\infty} E_n$ *is closed:*	AT	AF	(Nei)
Set $U \setminus K$ *is open:*	(AT)	AF	Nei
Set $U \cup K$ *is open:*	AT	AF	(Nei)
Set $E \cap K$ *is closed:*	(AT)	AF	Nei

Soln: None of the above stmts is *Always False*.

With respective centers at $(\pm 3, 0)$, radius-5 circles A_+ and A_- are each path-connected. Yet intersection $A_+ \cap A_-$ is doubleton $\{(0, \pm 4)\}$, which is not connected [hence certainly not path-connected].

A union of *finitely* many closed-sets is closed, but an ∞ union could be neither open nor closed. In $\mathbb{R}$, for example, unioning closed intervals $\bigcup_{n=1}^{\infty} [5,\ 8-\frac{1}{n}]$ produces half-open $[5, 8)$, which neither $\mathbb{R}$-open nor $\mathbb{R}$-closed.

Set-difference $U \setminus K$ is $U \cap K^c$, an intersection of finitely-many open sets, hence is open.

A union $U \cup K$ of open-with-closed might be neither. Eg., in $\mathbb{R}$ note $(6, 8) \cup [5, 7]$ is $[5, 8)$; not open nor closed.

Intersection $E \cap K$ is closed, since the intersection of an *arbitrary* collection (not just finitely-many) of closed-sets is closed.

20 h Cross-ratio $[z, 2+\mathbf{i}, 4\mathbf{i}, 3] = \frac{\mathtt{a}z+\mathtt{b}}{\mathtt{c}z+\mathtt{d}}$, where

a= ________, b= ________, c= ________, d= ________.

CR. Setting $q_0 := 2+\mathbf{i}$, $q_1 := 4\mathbf{i}$ and $q_\infty := 3$, we have that $[z, q_0, q_1, q_\infty]$ equals

$$\frac{[z - q_0][q_1 - q_\infty]}{[z - q_\infty][q_1 - q_0]} = \frac{[z - q_0][4\mathbf{i}-3]}{[z - q_\infty][3\mathbf{i}-2]}.$$

So $\mathtt{a} = 4\mathbf{i}-3$ and $\mathtt{b} = [2+\mathbf{i}][3-4\mathbf{i}] = 10-5\mathbf{i}$. Further, $\mathtt{c} = 3\mathbf{i}-2$. and $\mathtt{d} = 3\cdot[2-3\mathbf{i}] = 6-9\mathbf{i}$. ♦

OYOP: *In grammatical English* ***sentences****, write your essay on every* 2^{nd} *line* (usually)*, so I can easily write between the lines.*

V3: For <u>*reals*</u> S, α, β, T, consider equation

$$\dagger: \quad S[x^2 + y^2] + \alpha x + \beta y + T = 0$$

in $\mathbb{R}\times\mathbb{R}$. Show that (†) describes a ***gen-circle*** [i.e, a *circle-or-line*; a ***generalized-circle***] IFF

$$*: \quad \alpha^2 + \beta^2 > 4ST.$$

Proof. CASE: $S=0$. Our two displays become

$$\dagger': \quad \alpha x + \beta y + T = 0 \quad \textit{and}$$
$$*': \quad \alpha^2 + \beta^2 > 0.$$

Then $(*')$ fails exactly when $(\alpha, \beta) = (0,0)$. And our $(\dagger')$ does not describe a gen-circle, since its soln-set is *either* $\mathbb{R}\times\mathbb{R}$ (if $T=0$), *or* is empty (if $T\neq 0$).

In contrast, when $(\alpha,\beta) \neq (0,0)$ then: Our (†) is a vertical line when $\beta = 0$, else is a line of slope $\frac{-\alpha}{\beta}$.

CASE: $S\neq 0$. Rewrite (†) as

$$[x^2 + y^2] + \tfrac{\alpha}{S}x + \tfrac{\beta}{S}y + \tfrac{T}{S} = 0.$$

Completing the square gives

$$\ddagger: \quad \left[x^2 + \tfrac{\alpha}{2S}\right]^2 + \left[y^2 + \tfrac{\beta}{2S}\right]^2 = \tfrac{\alpha^2}{[2S]^2} + \tfrac{\beta^2}{[2S]^2} - \tfrac{T}{S} = \tfrac{1}{[2S]^2}\cdot\left[\alpha^2 + \beta^2 - 4ST\right].$$

This describes a circle IFF RhS(‡) is positive, i.e, IFF (∗). In that case, the circle has center $\left(\frac{-\alpha}{2S}, \frac{-\beta}{2S}\right)$, with radius $\frac{1}{2S}\cdot\sqrt{\alpha^2 + \beta^2 - 4ST}$. ♦

End of Class-V

V1:	___ ___ ___	115pts
V2:	___ ___	70pts
V3:	___ ___	45pts
Total:	___ ___ ___	230pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor (or his colleague)."*

Ord: