Closure properties of the class of Uniform Sweeping-out transformations

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ABSTRACT. Uniform sweeping-out is closed under countable cartesian product and under inverse limits.

A measure-preserving transformation $(S\colon X, \mu)$ on a probability space is ***uniform sweeping-out*** if for any set A of positive mass and any ε there exists N such that: Any collection $\mathbb{K}$ of integers will satisfy

$$\mu\Big(\bigcup_{k\in\mathbb{K}} S^k A\Big) > 1-\varepsilon$$

if $\#\mathbb{K} \geq N$. Nat Friedman introduced this property in [F]. Our goal here is to affirmatively answer a question of Friedman by showing that the class of uniform sweeping-out transformations is closed under countable cartesian product. The proof is a second application of the conditional expectation argument of [K] followed by a counting argument. I am indebted to Nat Friedman and Dan Rudolph who pointed out that uniform sweeping-out has a "mixinglike" characterization. This provoked the "lightly-mixing" characterization which is (C0) below and suggested dusting off the argument which shows that the class of lightly-mixing maps is closed under cartesian product. It is not known (in the category of weak-mixing transformations) whether uniform sweeping-out is implied by the existence of a dense family of sets A each of which sweeps-out uniformly.

Our cartesian product result appears now, rather than in 1988 when it was done, because it is now known that uniform sweeping-out is strictly weaker than mixing. (That mixing implies u.s.o appears in [F].) Terry Adams [A] has recently announced that the lightly-mixing example of [F,K] has the stronger uniform sweeping-out property. Yet it is not mixing; indeed, not even partial-mixing.

A "lightly-mixing" characterization. Each of the following two properties is equivalent to uniform sweeping-out. For a $\beta \in [0,1]$ let $\mathrm{Indices}_\beta(A,B)$ represent the set of indices k satisfying $\mu(S^kA\cap B) \leq \beta$. The function $\mathrm{Zero}(\cdot,\cdot)$ below is a uniform bound on the cardinality of such a

*Partially supported by

set of indices. In the sequel δ and ε are numbers in $(0,1)$. The phrase $a \coloneqq b$ means that the expression b *defines* the (new) symbol a.

(C0) *For any set A of positive mass, any ε, there exists $M \coloneqq \mathrm{Zero}(A,\varepsilon)$ such that for any set B with $\mu(B) \geq \varepsilon$:*

$$\#\mathrm{Indices}_0(A,B) < M.$$

If S satisfies (C0) then, given any collection $\mathbb{K}$ with $\#\mathbb{K} \geq M$, let B be the complement of the union $\bigcup_{k\in\mathbb{K}} S^k A$. Were $\mu(B)$ at least ε we could apply (C0) to obtain a contradiction. Hence any M iterates of A sweep out more than $1-\varepsilon$ of the space.

A similar argument shows the converse, that uniform sweeping-out implies (C0). ♦

(C1) *For any set A of positive mass, any ε, there exists $M \coloneqq \mathrm{Small}(A,\varepsilon)$ and positive number $\delta \coloneqq \mathrm{Size}(A,\varepsilon)$ such that for any set B with $\mu(B) \geq \varepsilon$:*

$$\#\mathrm{Indices}_\delta(A,B) < M.$$

Evidently (C1) implies (C0). Conversely, fix ε, A, and $M \coloneqq \mathrm{Zero}(A,\varepsilon/2)$. Set $\delta \coloneqq \varepsilon/2M$. Fix any B of mass at least ε. Suppose there were a collection $\mathbb{K}$, $\#\mathbb{K} = M$, of indices k such that $\mu(S^k A \cap B) \leq \delta$. Then the difference set

$$B' \coloneqq B \sim \bigcup_{k\in\mathbb{K}} S^k A$$

has mass at least $\varepsilon/2$. Yet $\mathrm{Indices}_0(A,B')$ contains $\mathbb{K}$, a contradiction. We conclude that the quantity $\mathrm{Small}_\delta(A,\varepsilon)$ is dominated by M. ♦

Remark. For the properties above, to emphasize the dependence on the transformation S we may write $\mathrm{Zero}(A,\varepsilon;S)$, etc.

The class of uniform sweeping-out transformations is evidently closed under powers and roots since

$$\mathrm{Zero}(A,\varepsilon;S^n) \leq \mathrm{Zero}(A,\varepsilon;S) \leq |n| \cdot \mathrm{Zero}(A,\varepsilon;S^n)$$

for any non-zero integer n.

Cartesian Product. Fix $(S\colon X,\mu)$ and $(T\colon \widehat{X},\widehat{\mu})$, two uniform sweeping-out transformations. Our goal is to show that $S\times T$ is uniform sweeping-out by showing it to satisfy (C0). Fix some set $\mathbf{V} \subset X \times \widehat{X}$ of positive mass. We shall compute an upper bound for

$$\mathrm{Zero}(\mathbf{V}, 2\varepsilon; S\times T)$$

in terms of $\mathrm{Small}(\ ,\varepsilon;S)$ and $\mathrm{Small}(\ ,\varepsilon;T)$.

Given a point $z \in X$ let $\mathbf{V}_z$ denote the cross-section of $\mathbf{V}$ above z; thus $\mathbf{V}_z$ is the subset of $\widehat{X}$ such that $\{z\}\times\mathbf{V}_z$ equals $\big[\{z\}\times\widehat{X}\big]\cap\mathbf{V}$. By standard measurability arguments, the following holds for μ-a.e. z. Set $\widehat{A} \coloneqq \mathbf{V}_z$. Then for any positive $\widehat{\delta}$ the set

$$V \coloneqq \big\{x \mid \widehat{\mu}(\mathbf{V}_x \triangle \widehat{A}) \leq \widehat{\delta}\big\} \tag{1}$$

has positive μ-mass. Consider z and $\widehat{A}$ as henceforth fixed. Define the quantities

$$\widehat{M} \coloneqq \mathrm{Small}(\widehat{A}, \varepsilon; T) \quad \text{and} \quad \widehat{\delta} \coloneqq \mathrm{Size}(\widehat{A}, \varepsilon; T).$$

For this $\widehat{\delta}$, define V as in (1). Finally, set

$$M \coloneqq \mathrm{Small}(V, \varepsilon; S) \quad \text{and} \quad \delta \coloneqq \mathrm{Size}(V, \varepsilon; S).$$

Wishing to establish (C0) for $S \times T$, it suffices to show that

$$\mathrm{Zero}(\mathbf{V}, 2\varepsilon; S \times T) \leq M + \widehat{M}/\delta.$$

Fix any set $\mathbf{W} \subset X \times \widehat{X}$ with mass at least 2ε. Set

$$W \coloneqq \{x \mid \widehat{\mu}(\mathbf{W}_x) \geq \varepsilon\}$$

and note that $\mu(W) \geq \varepsilon$ follows by a Fubini argument. Define a function $f\colon \mathbb{Z} \to [0,1]$ by

$$f(k) \coloneqq \mu\big\{x \in W \bigm| \widehat{\mu}(T^k \widehat{A} \cap \mathbf{W}_x) \leq \widehat{\delta}\big\}.$$

This function measures the probability that a fiber $\mathbf{W}_x$ has k in its bad set $\mathrm{Indices}_{\widehat{\delta}}(\widehat{A}, \mathbf{W}_x)$. Let $\mathbf{1}[\cdot]$ denote the Dirac function where $\mathbf{1}[\textbf{true}] = 1$ and $\mathbf{1}[\textbf{false}] = 0$. By Fubini, the sum $\sum_{k\in\mathbb{Z}} f(k)$ equals

$$\begin{aligned}\sum_k \int_W \mathbf{1}\big[\widehat{\mu}(T^k \widehat{A} \cap \mathbf{W}_x) \leq \widehat{\delta}\big]\, d\mu(x) &= \int_W \sum_k \mathbf{1}\big[\widehat{\mu}(T^k \widehat{A} \cap \mathbf{W}_x) \leq \widehat{\delta}\big]\; d\mu(x)\\ &\leq \int_W \#\mathrm{Indices}_{\widehat{\delta}}(\widehat{A}, \mathbf{W}_x)\, d\mu(x) \leq \mu(W) \cdot \widehat{M}.\end{aligned}$$

This yields the inequality

$$\sum_{k\in\mathbb{Z}} f(k) \leq \widehat{M}$$

whose usefulness arises from the fact that although $f(\cdot)$ depends on the set $\mathbf{W}$, the bound $\widehat{M}$ does not.

Counting the set of bad k. Suppose k is such that $[S\times T]^k \mathbf{V} \cap \mathbf{W}$ has zero mass. For μ-a.e. x then $\widehat{\mu}\big(T^k(\mathbf{V}_{S^{-k}x}) \cap \mathbf{W}_x\big)$ equals zero. Thus if $x \in S^k V$ then

$$\widehat{\mu}(T^k \widehat{A} \cap \mathbf{W}_x) \leq \widehat{\delta} \tag{2}$$

by (1). In particular, (2) holds for every $x \in S^k V \cap W$. Thus $f(k) \geq \mu(S^k V \cap W)$. This last quantity will exceed δ if k is chosen outside of $\mathbb{K} \coloneqq \mathrm{Indices}_\delta(V, W; S)$. As a consequence

$$\#\big(\mathrm{Indices}_0(\mathbf{V}, \mathbf{W}; S \times T) \sim \mathbb{K}\big) \leq \sum_{k\in\mathbb{Z}} \frac{f(k)}{\delta}.$$

Since the righthand quantity is dominated by $\widehat{M}/\delta$ we may conclude that

$$\#\mathrm{Indices}_0(\mathbf{V}, \mathbf{W}; S \times T) \leq \#\mathbb{K} + \widehat{M}/\delta \leq M + \widehat{M}/\delta$$

as desired. ♦

Countable Cartesian Products

In order to pass from finite to countable cartesian products we need to show that the class "Uniform Sweeping-out" is closed under inverse limits.

Given $(T\colon X,\mu)$ and a factor algebra $\mathcal{F}$ recall that the ***conditional probability*** function $\mathcal{P}[\,\cdot\,|\mathcal{F}]$ is canonically defined by the equality

$$\int_F \mathcal{P}[B|\mathcal{F}](x)\,d\mu(x) = \mu(B\cap F)$$

for all $F\in\mathcal{F}$ and measurable B.

Inverse limit Lemma. *Given $(T\colon X,\mu)$ and an increasing tower $\mathcal{F}_1\subset\mathcal{F}_2\subset\ldots$ of factor algebras whose join is the entire σ-algebra. Then*

$$T|_{\mathcal{F}_n} \text{ uniform sweeping-out for all } n \quad\Longrightarrow\quad T \text{ uniform sweeping-out.}$$

Proof. Fix ε and a set A of positive mass. Pick $\mathcal{F}\in\{\mathcal{F}_n\}_n$ sufficiently far out in the sequence that A is nearly $\mathcal{F}$-measurable: Choose it so that $\mu(F)$ is positive, where

$$F := \big\{x \;\big|\; \mathcal{P}[A|\mathcal{F}](x) > 1-\varepsilon\big\}.$$

Let N be the constant arising from the uniform sweeping-out of $T|_{\mathcal{F}}$; thus for any collection $\#\mathbb{K}\geq N$ of integers, $\mu(\mathbf{F})>1-\varepsilon$ where $\mathbf{F}$ denotes the union $\bigcup_{k\in\mathbb{K}} T^kF$. Let $\mathbf{A}:=\bigcup_{k\in\mathbb{K}} T^kA$. Consider a point $x\in\mathbf{F}$, say, $x\in T^kF$. Then

$$\mathcal{P}[\mathbf{A}|\mathcal{F}](x)\geq \mathcal{P}[T^kA|\mathcal{F}](x) = \mathcal{P}[A|\mathcal{F}](T^{-k}x) > 1-\varepsilon$$

where the last inequality follows from the definition of F. Consequently

$$\begin{aligned}\mu\Big(\bigcup_{k\in\mathbb{K}} T^kA\Big) = \int \mathcal{P}[\mathbf{A}|\mathcal{F}]\,d\mu &\geq \int_{\mathbf{F}} \mathcal{P}[\mathbf{A}|\mathcal{F}]\,d\mu\\ &\geq \int_{\mathbf{F}} 1-\varepsilon\,d\mu\\ &= \mu(\mathbf{F})\cdot(1-\varepsilon) > (1-\varepsilon)^2 > 1-2\varepsilon.\end{aligned}$$

Thus any N iterates of A sweep out all but 2ε of the space. ♦

References

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