Ord: ___ ___

Differential Eqns
MAP2302 3228

U-Class

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U1: *Show no work.*

a A multivariate polynomial, where each monomial has the same degree, is circle

monogamous **atrocious** **gregarious**
monic **expialadocious** **homogeneous**
manic **unitary** **Unitarian** **utilitarian**

b With $G(x) := \sin(\sin(x))$, a soln to $y'' - y = G$ is $y := f \circledast G$,

where $f(x) = \frac{1}{2}\left[e^{x} - e^{-x}\right]$.

Soln: The general ZeroTar soln f to $f'' - f = 0$ is $f(x) = \alpha e^{x} + \beta e^{-x}$. We need numbers α, β so that $f(0) = 0$ and $f'(0) = \frac{1}{1}$. So $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$.

c Fncs $x(t)$ and $y(t)$ satisfy this system of DEs,

$$\begin{aligned} x' + \ x - 3y &= 0, \\ y' + 6x - 8y &= 0. \end{aligned}$$

It can be written as $Y' = M \cdot Y$,

where $Y := \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ and M is matrix $\begin{bmatrix} -1 & 3 \\ -6 & 8 \end{bmatrix}$.

Characteristic poly of M is $\wp_M(z) = z^2 - 7z + 10$

A soln has $x(t)$ a linear combination of $e^{\alpha t}$ and $e^{\beta t}$ for *numbers* $\alpha = 2$ and $\beta = 5$.

Soln: DE-System. Char-poly $\wp_M(z)$ is

$$\begin{aligned} \operatorname{Det}(M - z\mathbf{I}) &= [-1 - z] \cdot [8 - z] - 3 \cdot [-6] \\ &= z^2 - 7z + 10 \stackrel{\text{note}}{=\!=} [z - 2] \cdot [z - 5]. \end{aligned}$$

d Op $\mathsf{L}(y) := 3t^2y'' + 5ty' - y$ is equidim'nal. The gen.soln to $\mathsf{L}(y)=0$ is $y(t) = \alpha\cdot\ t^{1/3}\ + \beta\cdot\ t^{-1}$.

EquiDimensional Soln: What real numbers $\mathbf{r}$ satisfy $\mathsf{L}(t^{\mathbf{r}}) = 0$? Note that $\mathsf{L}(t^{\mathbf{r}}) = t^{\mathbf{r}} \cdot q(\mathbf{r})$, where $q(z) := 3z^2 + [5-3]z - 1 = 3z^2 + 2z - 1$.

Here, $\text{Discr}(q) = 2^2 - 4{\cdot}3{\cdot}[-1] = 4^2$. So the q-roots are $\frac{1}{2\cdot 3}\big[-2 \pm 4\big]$, ie. $1/3$ and -1. Hence the general y with $\mathsf{L}(y)=0$ is $y(t) = \alpha{\cdot}t^{1/3} + \beta{\cdot}t^{-1}$.

Aside. For cut&paste:

```
3 t^2 y''(t) + 5 t y'(t) - y(t)  =  0
```

□

e) Inverse of $\mathsf{C} := \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, is $\mathsf{C}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$.

Conjugating $\mathsf{W} := \begin{bmatrix} 9 & -5 \\ 10 & -6 \end{bmatrix}$ by C

gives diagonal matrix $\mathsf{D} := \mathsf{C}^{-1}\mathsf{W}\mathsf{C} = \begin{bmatrix} -1 & \\ & 4 \end{bmatrix}$.

Thus the $(2,1)$-entry of $\mathsf{e}^{t\mathsf{W}}$ is $-2\mathsf{e}^{-t} + 2\mathsf{e}^{4t}$.

Soln: Firstly, $\mathsf{C}^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$, since $\text{Det}(\mathsf{C}) = -1$. Multiplying, $\mathsf{D} = \begin{bmatrix} -1 & \\ & 4 \end{bmatrix}$. With placeholders α, β, compute

$$*: \quad \mathsf{C} \cdot \begin{bmatrix} \alpha & \\ & \beta \end{bmatrix} \cdot \mathsf{C}^{-1} = \begin{bmatrix} -\alpha + 2\beta & \alpha - \beta \\ -2\alpha + 2\beta & 2\alpha - \beta \end{bmatrix}.$$

[We only need the the lower-LHand entry, but computing the entire matrix allows to check that: *Plugging* $\alpha := -1$ *and* $\beta := 4$ *in to* $(*)$, *replaces the placeholder matrix by* D, *hence produces* W.] In $(*)$, replacing the placeholder by diagonal matrix

$$\mathsf{e}^{\mathsf{D}t} = \begin{bmatrix} \mathsf{e}^{-t} & \\ & \mathsf{e}^{4t} \end{bmatrix},$$

produces $\mathsf{e}^{\mathsf{W}t}$. With $\alpha := \mathsf{e}^{-t}$ and $\beta := \mathsf{e}^{4t}$, the $(*)$ lower-LHand entry is $-2\mathsf{e}^{-t} + 2\mathsf{e}^{4t}$.

Aside. For cut&paste:

```
inverse({{1, 1},{2,1}})
[inverse({{1, 1},{2,1}})] . {{9, -5},{10,-6}} . {{1, 1},{2,1}}
{{1, 1}, {2, 1}} . {{A, 0}, {0, B}} . {{-1, 1}, {2, -1}}
```

□

f $\mathbb{R}$-matrices $\mathsf{U}, \mathsf{G}, \mathsf{R}$ are $3{\times}3$, with U invertible and R nilpotent. [Use $\mathbf{I}$ for the $3{\times}3$ identity matrix.]

Matrix URU^{-1} *is nilpotent*: **(AT)** AF Nei

Each entry of $\mathsf{e}^{t\mathsf{R}}$ *is a polynomial*: **(AT)** AF Nei

Matrix e^{R} *is nilpotent*: AT **(AF)** Nei

Matrix e^{R} is invertible, hence never nilpotent.

R^2 *is the zero-matrix*: AT AF **(Nei)**

Zero-matrix is "yes", and $\begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}$ is "no".

Matrix $\mathsf{e}^{[\mathsf{G}+\mathbf{I}]\mathsf{G}}$ *equals* $\mathsf{e}^{\mathsf{G}} \cdot \mathsf{e}^{\mathsf{G}^2}$: **(AT)** AF Nei

Note $[\mathsf{G}+\mathbf{I}]\mathsf{G} = \mathsf{G}^2 + \mathsf{G} = \mathsf{G} + \mathsf{G}^2$. And $\mathsf{G} \leftrightarrows \mathsf{G}^2$.

Matrix $\mathsf{e}^{[\mathsf{G}^2]}$ *equals* $\left[\mathsf{e}^{\mathsf{G}}\right]^2$: AT AF **(Nei)**

For the zero-matrix, $\mathbf{0}$, note $\mathsf{e}^{[\mathbf{0}^2]} = \mathbf{I} = \mathbf{I}^2 = [\mathsf{e}^{\mathbf{0}}]^2$; so "yes". But for identity-matrix: $\mathsf{e}^{[\mathbf{I}^2]} = \mathsf{e}^{\mathbf{I}} = \mathsf{e} \cdot \mathbf{I}$. Yet $[\mathsf{e}^{\mathbf{I}}]^2 = [\mathsf{e} \cdot \mathbf{I}]^2 = \mathsf{e}^2 \cdot \mathbf{I}$. As $\mathsf{e}^2 \neq \mathsf{e}$, this shows "no".

U2: *Show no work.*

Suppose $y(0) = -2$, $y'(0) = 5$, $y''(0) = 2$. Then $\mathcal{L}\big(y^{(3)} + y^{(2)} + 3y\big)(s)$ equals $\big[[B(s)\cdot\widehat{y}(s)] + C(s)\big]$ for **polynomials**

$B(s)=$ $s^3 + s^2 + 3$

and $C(s)=$ $2s^2 - 3s - 7$.

Soln: Polys. Our $B(s)$ is the aux-poly of the DOp. Summing over natnum pairs (j,k), recall formula

$$\mathcal{L}\big(y^{(N)}\big)(s) \;=\; s^N\cdot\widehat{y}(s) - \Big[\sum_{j+k=N-1} y^{(k)}(0)\cdot s^j\Big].$$

Applied at $N = 0, 1, 2$, this says our $C(s)$ equals ...

Aside. For cut&paste:

```
LaplaceTransform[y'''(t) + y''(t) + 3y(t), t, s]

LaplaceTransform{y'''(t) + y''(t) + 3y(t) = 0,  y(0)=-2, y'(0)=5, y''(0)=2}
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□

U1:	___ ___ ___	145pts
U2:	___ ___	20pts
Total:	___ ___ ___	165pts