Guaranteeing triangles in a graph

J.L.F. King
University of Florida, Gainesville FL 32611-2082, USA
9 April, 2018 (at *08:44*)

Entrance. Two proofs from Miklós Bóna's text. Below, ***graph*** means a finite simple graph.

When the relevant graph is evident from context, let $\widehat{\mathbf{u}} := \mathrm{Deg}(\mathbf{u})$ be the degree of vertex $\mathbf{u}$.

1: Lemma (Bóna:11.7). *Suppose graph $H=(\mathbb{V},\mathbb{E})$ has $2\mathbf{m}$ vertices, where $\mathbf{m} \geqslant 2$, and has at least $[1+\mathbf{m}^2]$ edges. Then H admits a triangle.* ♢

Rem. The result is "true" for $\mathbf{m}=1$, since a 2-vertex simple graph cannot have $\geqslant[1+\mathbf{m}^2] = 2$ edges. (For $\mathbf{m}=0$, it even *more* vacuously true.) □

Proof. For $\mathbf{m}=2$, our H has subgraph $K_4 \smallsetminus \{\text{edge}\}$, which has a triangle (indeed, two triangles).

Fix $\mathbf{m}\geqslant 3$ and an adjacent-pair $\mathbf{u}$—$\mathbf{v}$ of vertices. WLOG, $\mathbf{u}$ and $\mathbf{v}$ have no neighbor in common. Thus $\boxed{2\mathbf{m} \geqslant \widehat{\mathbf{u}} + \widehat{\mathbf{v}}}$. Delete $\mathbf{u}$ and $\mathbf{v}$ and their edges, to produce subgraph $H'=(\mathbb{V}',\mathbb{E}')$. This operation deleted $\widehat{\mathbf{u}}+\widehat{\mathbf{v}} - 1$ edges, since $\mathbf{u}$—$\mathbf{v}$. Hence

$$|\mathbb{V}'| = 2\cdot[\mathbf{m}-1] \quad \textit{and}$$
$$|\mathbb{E}'| \geqslant [1+\mathbf{m}^2] - [2\mathbf{m}-1] \overset{\text{note}}{=\!=} 1 + [\mathbf{m}-1]^2.$$

By induction on $\mathbf{m}$, then, H' admits a triangle. ♦

2: Proposition. *Let d denote the minimum vertex-degree of graph $R=(\mathbb{V},\mathbb{E})$ with $N\geqslant 1$ vertices and $E := |\mathbb{E}|$ edges. Have R' be R but with a min-deg vertex (and its edges) deleted. Then*

¥: $$|\mathbb{E}'| \geqslant |\mathbb{E}| \cdot \tfrac{N-2}{N},$$

with equality IFF R is vertex-regular ($\forall \mathbf{u}$: $\widehat{\mathbf{u}} = d$). ♢

Proof. Inequality (¥) follows from observing that

$$\frac{E-E'}{E} = \frac{d}{E} = \frac{2d}{2E} \leqslant \frac{2d}{Nd} = \frac{2}{N},$$

since $2E = \sum_{\mathbf{u}\in\mathbb{V}} \widehat{\mathbf{u}} \geqslant Nd$, with equality IFF vertex-regular. ♦

3: Theorem (Bóna:11.8). *The H of Lemma 1 admits at least $\mathbf{m}$ triangles.* ♢

Rem. As in the previous REMARK, this is vacuously true at $\mathbf{m}=1$ and $\mathbf{m}=0$. (For $\mathbf{m}=0$ it is even truer. Not only is the hypothesis vacuous, but the conclusion holds.) □

Pf. Since $K_4 \smallsetminus \{\text{edge}\}$ has two triangles, WLOG $\mathbf{m} \geqslant 3$.

Courtesy (1), we can fix the vertices $\mathbf{u},\mathbf{v},\mathbf{w}$ of a triangle, $\triangle$. The number of other vertices is $[2\mathbf{m}-3]$, so define $x \in \mathbb{Z}$ by

$$x + [2\mathbf{m}-3] := \begin{bmatrix}\text{Number of \textbf{\textit{connector}} edges, con-} \\ \text{necting } \triangle \text{ to the ``other'' vertices}\end{bmatrix}.$$

Notice that if $x \geqslant 0$, then these "other" vertices give us at least x many triangles formed with some two of $\mathbf{u},\mathbf{v},\mathbf{w}$.

CASE: $\mathbf{m}-1 \leqslant x$ Then the $\mathbf{m}-1$ "other" triangles, along with $\triangle$, give us the requisite $\mathbf{m}$ triangles.

CASE: $1 \leqslant x < \mathbf{m}-1$ Let R denote the induced-subgraph of the other $[2\mathbf{m}-3]$ vertices. The number of connectors is

$$x + [2\mathbf{m}-3] \leqslant [\mathbf{m}-2] + [2\mathbf{m}-3] = 3\mathbf{m}-5\,.$$

Hence the number of R-edges is *at least*

$$\dagger\text{:} \quad [1+\mathbf{m}^2] - [3\mathbf{m}-5] - 3 = \mathbf{m}^2 - 3\mathbf{m} + 3\,.$$

Let R' be R but with a min-deg vertex deleted; so $|\mathbb{V}_{R'}| = 2\mathbf{m}-4 = 2[\mathbf{m}-2]$. If we could establish

$$*\text{:} \qquad |\mathbb{E}_{R'}| \overset{?}{>} [\mathbf{m}-2]^2\,,$$

then H would have at least $1 + x + [\mathbf{m}-2]$ triangles, i.e, at least $\mathbf{m}$ triangles (courtesy the induction).

By the Proposition, $|\mathbb{E}_{R'}| \geqslant |\mathbb{E}_R| \cdot \frac{2\mathbf{m}-5}{2\mathbf{m}-3}$. So ISTS

$$|\mathbb{E}_R| \cdot [2\mathbf{m}-5] \overset{?}{>} [2\mathbf{m}-3] \cdot [\mathbf{m}-2]^2\,.$$

Courtesy (†), then, assertion

$$\ddagger\text{:} \quad [\mathbf{m}^2 - 3\mathbf{m} + 3] \cdot [2\mathbf{m}-5] \overset{?}{>} [2\mathbf{m}-3] \cdot [\mathbf{m}-2]^2$$

suffices. "Easily", LhS(‡) − RhS(‡) $= \mathbf{m}-3$. Consequently, $\mathbf{m} \geqslant 4$ implies (‡).

For $\mathbf{m}=3$, failure of ($*$), forces every preceding inequality to be an equality. So graph R, which has $2\mathbf{m}-3 = 6-3 = 3$ vertices, would have to have exactly

$$\mathbf{m}^2 - 3\mathbf{m} + 3 \overset{\text{by (†)}}{=\!=} 9 - 9 + 3 = 3$$

edges. Thus R is itself a triangle, and consequently indeed has $\mathbf{m}-2 = 3-2 = 1$ triangles, even though R' has only 2 vertices and hence no triangles.

CASE: $x \leqslant 0$ Now R has at least

$$[1+\mathbf{m}^2] - [x + [2\mathbf{m}-3] + 3] = [\mathbf{m}-1]^2 - x$$

edges. If strict $x < 0$, then $|\mathbb{E}_R| \geqslant 1 + [\mathbf{m}-1]^2$. So adjoining, say, vertex $\mathbf{u}$ to R, gives a graph, R^+, with $2 \cdot [\mathbf{m}-1]$ vertices and at least $1 + [\mathbf{m}-1]^2$ edges. Thus R^+ has at least $\mathbf{m}-1$ triangles distinct from $\triangle$.

UPSHOT: WLOGenerality $x=0$, and the number of connectors is $2\mathbf{m}-3 \overset{\text{note}}{>} 0$. So *some* vertex of $\mathbf{u},\mathbf{v},\mathbf{w}$ has a connector to R; say vertex $\mathbf{w}$. Adjoining to R, vertex $\mathbf{w}$ and its connectors, again gives a graph with $2 \cdot [\mathbf{m}-1]$ vertices and at least $1 + [\mathbf{m}-1]^2$ edges. ♦

Filename: `Problems/GraphTheory/triangles-in-a-graph.latex`
As of: *Friday 06Apr2018.* Typeset: *9Apr2018 at 08:44.*