Proof of the Triangle Inequality from Cauchy-Schwarz

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1: Half-of the △-Inequality Thm. *Fix a natnum N. For all $\mathbf{u},\mathbf{w} \in \mathbb{R}^N$, then,*

$$*: \qquad \|\mathbf{u}+\mathbf{w}\| \ \le \ \|\mathbf{u}\| + \|\mathbf{w}\|\,.$$

We'll use this tool.

2: Lemma. *Consider reals α and β with*

$$\dagger: \qquad \alpha^2 \ \le \ \beta^2\,.$$

If β is non-negative, then

$$\ddagger: \qquad \alpha \ \le \ \beta\,.$$

Proof of (??). Fix vectors $\mathbf{u},\mathbf{w} \in \mathbb{R}^N$. By defn of norm, $\|\mathbf{u}+\mathbf{w}\|^2$ equals $\langle \mathbf{u}+\mathbf{w}, \mathbf{u}+\mathbf{w}\rangle$. So the bilinearity of inner-product yields that

$$\begin{aligned} \|\mathbf{u}+\mathbf{w}\|^2 \ &= \ \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{u},\mathbf{w}\rangle + \langle \mathbf{w},\mathbf{u}\rangle + \langle \mathbf{w},\mathbf{w}\rangle \\ &\le \ \langle \mathbf{u},\mathbf{u}\rangle + \|\mathbf{u}\|\|\mathbf{w}\| + \|\mathbf{w}\|\|\mathbf{u}\| + \langle \mathbf{w},\mathbf{w}\rangle\,, \end{aligned}$$

by the C-S Inequality used twice. Rewriting,

$$\|\mathbf{u}+\mathbf{w}\|^2 \ \le \ \Big[\|\mathbf{u}\| + \|\mathbf{w}\|\Big]^2.$$

Can we apply our lemma, with $\alpha := \|\mathbf{u}+\mathbf{w}\|$ and $\beta := \|\mathbf{u}\| + \|\mathbf{w}\|$? Since β is a sum of norms, our β is non-negative. Hence the lemma *does* apply, yielding that $\alpha \le \beta$, ie, yielding $(*)$. ♦

Filename: `Problems/Algebra/LinearAlg/triangle-inequality.latex`
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