

Taylor's theorem in several guises

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Preliminaries. A polynomial $p()$ is an “ N -topped polynomial” if $\text{Deg}(p) < N$. So $x^2 - 2x$ and $x + \sqrt{7}$ are 3-topped, but $x^3 + x - 5$ is not. The set of N -topped polynomials is an N -dimensional VS.^{♥1}

Taylor polynomials

The setting is an open interval $J \subset \mathbb{R}$. Use $\text{Diff}^N = \text{Diff}^N(J \rightarrow \mathbb{R})$ for the set of N -times differentiable fncs. This is a superset of

$$\mathbf{C}^N = \mathbf{C}^N(J \rightarrow \mathbb{R});$$

those, whose N^{th} derivative is *continuous*.

For posint N , a point $Q \in J$ and $f \in \text{Diff}^{N-1}$, define

1a: “the N^{th} **Taylor polynomial** of f , centered at Q ”

to be the *unique* N -topped polynomial $p()$ whose zero^{th} through $[N-1]^{\text{st}}$ derivatives, at Q , agree with those of f . I.e, these N equations hold:

$$p(Q) = f(Q), \quad p'(Q) = f'(Q), \quad p''(Q) = f''(Q), \\ \dots, \quad p^{(N-1)}(Q) = f^{(N-1)}(Q).$$

One easily checks that $p(x)$ must be the RhS of (1b), below. That is,

$$1b: \mathbf{T}_N(x) = \mathbf{T}_{N,Q}^f(x) := \sum_{k \in [0..N)} \frac{f^{(k)}(Q)}{k!} \cdot [x - Q]^k$$

is the N^{th} Taylor polynomial of f .

^{♥1}Abbreviations: VS for *vector space*; FTC for *Fundamental Theorem of Calculus*; posint for *positive integer*.

Define the “the N^{th} *remainder term* of f ”, written $\mathbf{R}_{N,Q}^f$ or just \mathbf{R}_N , by

$$1c: \quad f(x) =: \mathbf{T}_N(x) + \mathbf{R}_N(x).$$

Properties of \mathbf{T}_N . Use $\mathbf{T}_{N,Q}[f]$ as a synonym for $\mathbf{T}_{N,Q}^f$, with the same convention for the \mathbf{R}_N operator and friends. Often times either the fnc, f , or the center-point, Q , is implicit, and we drop it from the notation.

For a scalar α and $f, g \in \text{Diff}^{N-1}$, note that

$$2: \quad \begin{aligned} \mathbf{T}_N[\alpha \cdot f] &= \alpha \cdot \mathbf{T}_N[f] \quad \text{and} \\ \mathbf{T}_N[f + g] &= \mathbf{T}_N[f] + \mathbf{T}_N[g]. \end{aligned}$$

I.e, \mathbf{T}_N is a *linear operator* from Diff^{N-1} to the VS of polynomials. Also, for $f \in \text{Diff}^N$,

$$3: \quad \mathbf{T}_N[f'] = [\mathbf{T}_N[f]]'$$

That is, \mathbf{T}_N commutes with differentiation; in symbols $\mathbf{T}_N \Leftrightarrow \mathbf{D}$. Letting \mathbf{I} denote the identity operator, we have that $\mathbf{I} = \mathbf{T}_N + \mathbf{R}_N$. Since \mathbf{I} is linear and commutes with \mathbf{D} , so is/does \mathbf{R}_N .

Courtesy FTC, integrating (3) gives

$$\begin{aligned} \int_Q^x \mathbf{T}_{N,Q}[f'](t) \cdot dt &= \mathbf{T}_{N,Q}[f](x) - \mathbf{T}_{N,Q}[f](Q) \\ &= \mathbf{T}_{N,Q}^f(x) - f(Q). \end{aligned}$$

There is a corresponding statement about the remainder term. □

Taylor's Theorem for \mathbb{R}

We produce two estimates^{♥2} of the remainder term. In the results below, $N \in \mathbb{Z}_+$.

4: **Lem.** Fix an $f \in \text{Diff}^N(J \rightarrow \mathbb{R})$. At each $x \in J$, function $q \mapsto \mathbf{T}_{N,q}^f(x)$ is differentiable, with value

$$4': \quad \frac{d}{dq} \mathbf{T}_{N,q}^f(x) = f^{(N)}(q) \cdot \frac{[x-q]^{N-1}}{[N-1]!}.$$

And $\frac{d}{dq} \mathbf{R}_{N,q}^f(x) = -\frac{d}{dq} \mathbf{T}_{N,q}^f(x)$. ◇

^{♥2}The TayThm-1 estimate, (5a), is called *Lagrange's form* of the N^{th} remainder term.

Our TayThm-2, (6a), is one of the many *Integral forms* of the remainder.

Proof. WLOG $x=7$. For natnum k and posint ℓ , define

$$A_k(q) := \frac{f^{(k)}(q)}{k!} \cdot [7 - q]^k, \quad \text{and}$$

$$B_\ell(q) := \frac{f^{(\ell)}(q)}{[\ell-1]!} \cdot [7 - q]^{\ell-1}.$$

Now $A_0(q) = f(q) \cdot 1$, so $\frac{d}{dq}A_0(q) = f'(q) = B_1(q)$. And for k positive, $\frac{d}{dq}A_k(q)$ equals

$$\frac{1}{k!} \left[f^{(k+1)}(q) \cdot [7 - q]^k + f^{(k)}(q) \cdot k \cdot [7 - q]^{k-1} \cdot [-1] \right]$$

$\stackrel{\text{note}}{=} B_{k+1}(q) - B_k(q).$

Thus $\frac{d}{dq}$ of $\mathbf{T}_{N,q}^f(7)$ is (notationally dropping the “ q ”)

$$[B_N - B_{N-1}] + [B_{N-1} - B_{N-2}] + \dots + [B_2 - B_1] + B_1.$$

Telescoping, this equals B_N , which is RhS(4'). \blacklozenge

5: TayThm-1. Consider an $f \in \text{Diff}^N(J \rightarrow \mathbb{R})$. For distinct points $Q, x \in J$, there exists a point $c = c_{N,Q,x,f}$ strictly between Q and x , such that

$$5a: \quad \mathbf{R}_{N,Q}^f(x) = \frac{1}{N!} f^{(N)}(c) \cdot [x - Q]^N.$$

I.e., $f(x) = \left[\sum_{k=0}^{N-1} f^{(k)}(Q) \cdot \frac{[x-Q]^k}{k!} \right] + f^{(N)}(c) \cdot \frac{[x-Q]^N}{N!}.$ \blacklozenge

Pf. WLOG $Q = 3$ and $x = 7$. Define $M \in \mathbb{R}$ by

$$5b: \quad \mathbf{R}_{N,3}(7) = M \cdot \frac{[7-3]^N}{N!}.$$

(Possible, since $[7 - 3]^N \neq 0$.) Create a function

$$5c: \quad \varphi(q) := \mathbf{R}_{N,q}(7) - M \cdot \frac{[7-q]^N}{N!}.$$

Our “multiplier” M was defined so that $\varphi(3) = 0$. And $\varphi(7) = 0 - M \cdot 0 = 0$. (We used that $[7 - 7]^N = 0^N$ is zero, since N is positive.) (Exer.: Why is $\mathbf{R}_{N,7}(7) = 0$?)

Rolle's thm now applies to φ on $[3, 7]$, asserting a point $c \in (3, 7)$ st. $\varphi'(c) = 0$. By Lemma 4,

$$0 = -f^{(N)}(c) \cdot \frac{[7-c]^{N-1}}{[N-1]!} - M \cdot [-1] \cdot \frac{[7-c]^{N-1}}{[N-1]!}.$$

Since $c \neq 7$, quantity $[7 - c]^{N-1}$ is not zero; so we may divide, to conclude that $0 = -f^{(N)}(c) + M.$ \blacklozenge

6: TayThm-2. For $f \in \mathbf{C}^N(J)$ and points $Q, x \in J$:

$$6a: \quad \mathbf{R}_{N,Q}^f(x) = \frac{1}{[N-1]!} \int_Q^x f^{(N)}(t) \cdot [x - t]^{N-1} dt. \quad \blacklozenge$$

Proof. WLOG $Q = 3$ and $x = 7$. Define this function

$$\theta(q) := \mathbf{T}_{N,q}^f(7).$$

This $\theta \in \mathbf{C}^1$, since $f \in \mathbf{C}^N$. Since $\theta(7) = f(7)$,

$$\mathbf{R}_{N,3}^f(7) \stackrel{\text{def}}{=} f(7) - \mathbf{T}_{N,3}^f(7)$$

$$*: \quad = \theta(7) - \theta(3) = \int_3^7 \theta'$$

by FTC, since θ' is continuous on J . And RhS(*) equals RhS(6a), courtesy Lemma 4. \blacklozenge

Aside: An incorrect formula. The 1st edition of Buck's text has (6a), but the the 3rd ed. (P.148, top) asserts that $\mathbf{R}_{N,Q}^f(x)$ equals

$$6b: \quad W_{N,Q}^f(x) := \frac{1}{[N-1]!} \int_Q^x f^{(N)}(t) \cdot [t - Q]^{N-1} dt.$$

Just because RhS(6b) differs from RhS(6a) doesn't mean that it is wrong; there are several formulae for \mathbf{R}_N ITOF integrals. The following example, however, indeed shows that $W_{N,Q}^f \neq \mathbf{R}_{N,Q}^f$.

Let $f := [z \mapsto z^3]$, $Q := 0$ and $N := 2$. Thus $\mathbf{T}_2(x)$ equals $0 + 0 \cdot x$, so $\mathbf{R}_2(x) = x^3$. Since $f''(t) = 3 \cdot 2 \cdot t$, our $W_2(x)$ equals

$$**: \quad \frac{1}{1!} \int_0^x 3 \cdot 2 \cdot t \cdot [t - 0] \cdot dt = 2t^3 \Big|_{t=0}^{t=x} = 2x^3.$$

So $W_2(1) = 2 \neq 1 = \mathbf{R}_2(1)$. In contrast, RhS(6a) equals $\int_0^x 3 \cdot 2 \cdot t \cdot [x - t] \cdot dt$, which equals

$$\left[x \cdot \int_0^x 3 \cdot 2 \cdot t \cdot dt \right] - \text{RhS(**)} = x \cdot 3x^2 - 2x^3.$$

And this, happily, equals $\mathbf{R}_2(x)$. \square

Applications of Taylor's theorem

Here are five uses.

7: Translating a polynomial. Suppose you wish to express polynomial

$$p(x) = C_0 + C_1x + \dots + C_{N-1}x^{N-1}$$

in form $\sum_{k \in [0..N)} B_k \cdot [x - 6]^k$. You *could* solve a system of N many linear eqns in N unknowns.

But p **equals** its N^{th} TayPoly (centered anywhere we want). So each B_k is the k^{th} coeff of $\mathbf{T}_{N,6}^p$. Thus B_k equals $p^{(k)}(6)/k!$. \square

8: Limits. Suppose $f \in \text{Diff}^2(\mathbb{R} \rightarrow \mathbb{R})$ and $\lim_{x \rightarrow \infty} f(x)$ exists in \mathbb{R} , and $\limsup_{x \rightarrow \infty} |f''(x)| < \infty$. Then $f'(x) \rightarrow 0$ as $x \nearrow \infty$. \diamond

Set-up for both proofs. WLOG $\boxed{*: \lim_{x \rightarrow \infty} f(x) = 0}$ and $|f''(x)| \leq 8$. \square

Pf. Fixing $\varepsilon > 0$, we will produce an $L \in \mathbb{R}$ for which:

$$\forall x \geq L: |f'(x)| \leq 5\varepsilon.$$

Consider TayPoly $\mathbf{T}_{2,x}^f(y) = f(x) + f'(x)[y - x]$ for $N=2$. The TayThm-1 gives that for each pair $x < y$ there is a point $c_{x,y} \in (x, y)$ for which

$$f(y) = f(x) + f'(x)[y - x] + \frac{f''(c_{x,y})}{2}[y - x]^2.$$

Solve for $f'(x)$. Since $|f''(c_{x,y})| \leq 8$,

$$|f'(x)| \leq \left| \frac{f(y) - f(x)}{y - x} \right| + \frac{8}{2} \cdot [y - x].$$

By $(*)$, there is $L \in \mathbb{R}$ st. $|f| \leq \varepsilon^2/2$ on $[L, \infty)$. Setting $y = x + \varepsilon$, we have that for each $x \geq L$:

$$\begin{aligned} |f'(x)| &\leq \left| \frac{\varepsilon^2}{y-x} \right| + 4 \cdot [y - x] \\ &\leq \frac{\varepsilon^2}{\varepsilon} + 4\varepsilon = 5\varepsilon. \end{aligned} \quad \diamond$$

Proof, barehands. Could $\beta := \liminf_{x \rightarrow \infty} f'(x) > 0$? No; for then the graph of f would grow at some minimum rate, thus could not have a horizontal asymptote. Hence $\beta \leq 0$. Similarly $\alpha \geq 0$, where $\alpha := \limsup_{x \rightarrow \infty} f'(x)$. The upshot: $\alpha \geq 0 \geq \beta$.

FTSOC, suppose $\alpha > \beta$. WLOG $\alpha > 0$; otherwise, replace f by $-f$. Set $\varepsilon := \alpha/3$.

Let a_1 be, say, the smallest non-negative “ x -value” such that $f(a_1) \geq 2\varepsilon$. Since

$$\alpha > 2\varepsilon > \varepsilon > \beta,$$

the following process never stops: Take b_n to be the smallest “ x ” exceeding a_n for which $f(b_n) \leq \varepsilon$. Take a_{n+1} to be the smallest value exceeding b_n for which $f(a_{n+1}) \geq 2\varepsilon$.

Let $J_n := [a_n, b_n]$. Each restriction $f'|_{J_n} \geq \varepsilon$, for $n = 1, 2, \dots$. By FTC,

$$\int_{b_n}^{a_n} f'' = f'(a_n) - f'(b_n) \geq 2\varepsilon - \varepsilon = \varepsilon.$$

So $\varepsilon \leq \int_{J_n} |f''| \leq 8 \cdot [b_n - a_n]$. Hence each $b_n - a_n$ dominates $\frac{\varepsilon}{8}$; thus the numbers $a_n, b_n \nearrow \infty$.

By $(*)$, then, $f(b_n) - f(a_n) \rightarrow 0$, as $n \nearrow \infty$. But

$$\begin{aligned} f(b_n) - f(a_n) &= \int_{a_n}^{b_n} f' \geq \int_{a_n}^{b_n} \varepsilon \\ &\geq [b_n - a_n] \varepsilon \geq \frac{8}{\varepsilon} \cdot \varepsilon = 8. \end{aligned} \quad \blacklozenge$$

N^{th} -derivative test for extrema. With $J \subset \mathbb{R}$ an open interval and $f \in \mathbf{C}^\infty(J \rightarrow \mathbb{R})$, suppose $Q \in J$ is a **critical point** for f , ie. $f'(Q) = 0$. We seek a test determining if f has a *strict local-max/min* at Q . To this end, define $\text{MinNZD}(f, Q)$ to be

$$9a: \quad \inf \left\{ k \in \mathbb{Z}_+ \mid f^{(k)}(Q) \neq 0 \right\} \stackrel{\text{note}}{\in} [1.. \infty]. \quad \square$$

9b: Min/Max Prop'n. With J, f, Q as above:

Suppose $N := \text{MinNZD}(f, Q)$ is in $[2.. \infty)$.

When N is...

... even: If $f^{(N)}(Q) > 0$ then f has a *strict-local-min*; else, f has a *strict-local-max*, at Q .

...odd: Then f has a strict-SignChange at Q . \diamond

Pf. Let $V := f^{(N)}(Q)$, which is not zero. Since $f^{(N)}$ is continuous at Q , there is a small open interval $I \ni Q$ for which $f^{(N)}|_I$ has the same sign as V . Our Prop'n will be established by (9c), below.

Let $\mathbf{T} := \mathbf{T}_{N,Q}^f$ and $\mathbf{R} := \mathbf{R}_{N,Q}^f$. Recall that $f^{(k)}(Q) = 0$, for each $k \in [1..N]$. For each x , then, $\mathbf{T}(x) = f(Q)$, and thus $\mathbf{R}(x) = f(x) - f(Q)$.

For $x \neq Q$, there is a point τ_x , between x and Q , with $\mathbf{R}(x) = \frac{1}{N!} f^{(N)}(\tau_x) \cdot [x - Q]^N$. Taking $x \in I$ forces $\tau_x \in I$, so

$f^{(N)}(\tau_x)$ has the same sign as V .

But $f(x) - f(Q) = \mathbf{R}(x)$, so the sign-fnc gives

9c:
$$\text{Sgn}(f(x) - f(Q)) = \text{Sgn}(V) \cdot [\text{Sgn}(x - Q)]^N,$$
 for each $x \in I$ with $x \neq Q$. \diamond

10: Lemma. Set $J := [-1, 1]$. Imagine a function $f \in \text{Diff}^3(J \rightarrow \mathbb{R})$ with

10a:
$$f(0) = 0 = f'(0).$$

10b:
$$f(-1) = 0 \quad \text{and} \quad f(1) = 1.$$

Then there exists $\tau \in J^\circ$ with $f'''(\tau) \geq 3$. \diamond

Counting degrees-of-freedom. What would it mean if each derivative $\{f^{(k)}\}_{k=3}^\infty$ were identically-zero? Then f is 3-topped polynomial (i.e, at most quadratic) and so comes from a 3-dim'al VS.

But (10a,10b) are *four* conditions, making it plausible that we cannot fulfill them when constrained to the VS of 3-topped polynomials. \square

Proof. ISTProduce points $\alpha, \beta \in J^\circ$ with

10c:
$$f^{(3)}(\alpha) + f^{(3)}(\beta) \geq 6.$$

Let $\mathbf{T} := \mathbf{T}_{3,0}^f$ and $\mathbf{R} := \mathbf{R}_{3,0}^f$. Define a number M by

$$\mathbf{T}(x) = f(0) + f'(0)x + \overbrace{\frac{f''(0)}{2}x^2}^M \stackrel{\text{by (10a)}}{\geq} M \cdot x^2.$$

Thus $\mathbf{T}(\pm 1) = M \cdot [\pm 1]^2 = M$. From (10b), then,

$$\begin{aligned} 1 &= f(1) - f(-1) = [M + \mathbf{R}(1)] - [M + \mathbf{R}(-1)] \\ &= \mathbf{R}(1) - \mathbf{R}(-1). \end{aligned}$$

By TayThm-1, $\exists \alpha \in (-1, 0)$ and $\exists \beta \in (0, 1)$ with

$$\begin{aligned} \mathbf{R}(1) &= \frac{1}{6} \cdot f^{(3)}(\beta) \cdot [+1 - 0]^3 = \frac{1}{6} \cdot f^{(3)}(\beta); \\ \mathbf{R}(-1) &= \frac{1}{6} \cdot f^{(3)}(\alpha) \cdot [-1 - 0]^3 = -\frac{1}{6} \cdot f^{(3)}(\alpha). \end{aligned}$$

This, using the previous display, yields (10c). \diamond

DiffyQ. A fnc $f \in \text{Diff}^2(\mathbb{R} \rightarrow \mathbb{R})$ with $f(7) = 0$ and $f'(7) = 0$, satisfies

‡:
$$f'' + f = \mathbf{0}. \quad [\text{I.e, the zero-fnc.}]$$

Prove that $f = \mathbf{0}$, using Taylor's thm, as follows:

First, show that $f \in \mathbf{C}^\infty$. Then, for a *fixed* $x_0 \in \mathbb{R}$, argue that $|f(x_0)|$ is as small as desired, by upper-bounding with Taylor-remainder terms from **Our Taylor's pamphlet** on the Teaching Page.

Proof. Let $\|\cdot\|$ mean $\|\cdot\|_{\text{sup}}$. Induction on $n \in \mathbb{N}$ shows that each $f^{(n)}$ is diff'able. Moreover

‡:
$$\forall k, \ell \in \mathbb{N}: \text{ If } k \equiv_4 \ell \text{ then } f^{(k)} = f^{(\ell)}.$$

As $f^{(2)}(7) = -f(7) = 0$ and $f^{(3)}(7) = -f'(7) = 0$, property (‡) tells us that each $\mathbf{T}_{n,7}^f$ is the zero-function. In particular, with \mathbf{R}_n denoting $\mathbf{R}_{n,7}^f$,

$$\forall n \in \mathbb{Z}_+: \quad \mathbf{R}_n(x_0) = f(x_0).$$

Upper bounds. Let J be the compact interval going from 7 to x_0 . Let $M_n := \|f^{(n)}|_J\|$. Define

$$\mathbf{M} := \text{Max}(M_0, M_1, M_2, M_3).$$

Then \mathbf{M} dominates each $\|f^{(n)}|_J\|$, courtesy (‡). From TayThm-2, then, $|f(x_0)|$ equals $|\mathbf{R}_n(x_0)|$ which equals

$$\begin{aligned} & \frac{1}{[n-1]!} \cdot \left| \int_7^{x_0} f^{(n)}(t) \cdot [x_0 - t]^{n-1} \cdot dt \right| \\ & \leq \frac{1}{[n-1]!} \int_7^{x_0} |f^{(n)}(t)| \cdot |x_0 - t|^{n-1} \cdot dt \\ & \leq \frac{\mathbf{M}}{[n-1]!} \int_7^{x_0} |x_0 - t|^{n-1} \cdot dt. \end{aligned}$$

Let $L := \text{Len}(J)$. Then

$$|f(x_0)| \leq \frac{\mathbf{M}}{[n-1]!} \int_7^{x_0} L^{n-1} \cdot dt = \mathbf{M} \cdot \frac{L^n}{[n-1]!}.$$

This last goes to zero, as $n \nearrow \infty$. ♦

Radius of Convergence

Series notations. Customs about how “series” is used in the context of “convergence of a series” are a bit strange. A “**series** \vec{e} ” is a sequence $\vec{e} = (e_k)_{k=0}^\infty$, but ^{♥3} where the word “series” hints to the reader our interest in its sum $\Sigma(\vec{e})$. This sum is the limit –when it exists– of the corresponding “partial-sum sequence” \vec{s} , where

$$11: \quad s_N := \sum_{k \in [0..N)} e_k$$

Use $\boxed{\vec{s} = \mathbb{P}\Sigma(\vec{e})}$ to indicate this partial-sum relation between sequences. Here, phrase “series \vec{e} is convergent” means that $\lim(\vec{s})$ exists and is finite. So $\Sigma(\vec{e}) := \lim(\vec{s})$.

To clarify, the n^{th} partial sum means the sum of the first n terms, regardless of the initial index. For example, suppose $\vec{b} = (b_\ell)_{\ell=5}^\infty$, and $\vec{e} = \mathbb{P}\Sigma(\vec{b})$. Then $e_3 = b_5 + b_6 + b_7$, and $e_0 = 0$.

Example: Let $\vec{b} := (k^2)_{k=1}^\infty$ and $\vec{a} := \mathbb{P}\Sigma(\vec{b})$. Then $a_n = \frac{1}{6} \cdot [2n^3 + 3n^2 + n]$. \square

12: Root-test lemma. Given a series $\vec{e} \subset \mathbb{C}$, define

$$*: \quad \Lambda := \limsup_{n \rightarrow \infty} \sqrt[n]{|e_n|} \stackrel{\text{note}}{\in} [0, +\infty].$$

If $\Lambda < 1$ then \vec{e} is an absolutely-convergent series.

If $\Lambda > 1$ then \vec{e} is “magnificently divergent” Not only $|e_n| \not\rightarrow 0$, but indeed $\limsup_{n \rightarrow \infty} |e_n| = +\infty$. \diamond

Proof. Let $a_n := |e_n|$,

CASE: When $\Lambda < 1$. ISTShow that \vec{a} is a convergent series. Pick ρ with $\Lambda < \rho < 1$. Take K large enough that $\sup_{n \geq K} \sqrt[n]{a_n} \leq \rho$. Hence $\sum_{n \geq K} a_n \leq \sum_{n \geq K} \rho^n < \infty$. And $\sum_{n \in [1..K]} a_n < \infty$.

CASE: When $\Lambda > 1$. Pick ρ with $1 < \rho < \Lambda$. By (*), the set $J := \{n \mid \sqrt[n]{a_n} > \rho\}$ is infinite. And each $n \in J$ has $a_n > \rho^n$. \diamond

^{♥3}The index will usually start at zero, but it doesn’t have to. The sequence \vec{e} might be $(e_k)_{k=24}^\infty$, or $(e_k)_{k=-5}^\infty$.

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is *eventually positive* if $\exists K \text{ s.t } \forall x \geq K: f(x) > 0$. Thus a degree- k poly,

$$f(x) := C_k x^k + \dots + C_1 x + C_0,$$

is eventually positive IFF f has positive leading-coeff, $C_k > 0$.

Power-series notation. A sequence $\vec{c} \subset \mathbb{C}$ and point $Q \in \mathbb{C}$ determine a **power series**

$$13a: \quad \text{PS}_{\vec{c}, Q}(z) := \sum_{n=0}^\infty c_n \cdot [z - Q]^n. \quad \square$$

From the notation we sometimes drop the the center of expansion, just writing $\text{PS}_{\vec{c}}$. This is especially true when the center of expansion is $0 \in \mathbb{C}$.

Use “PS” to abbreviate the phrase “power series”. Use McS to abbrev **Maclaurin Series**; a PS centered at $Q=0$. E.g $\text{McS}_{\vec{c}}(z) = \sum_{n=0}^\infty [c_n \cdot z^n]$.

Radius of Convergence. The set of $z \in \mathbb{C}$ for which RhS(13a) converges is called the “**set-of-convergence**”. We write it $\text{SoC}(\vec{c}, Q)$

It will turn out that the SoC comprises an open ball, possibly of radius 0 or ∞ , together with some of the points on the boundary of this ball. This open **ball of convergence** is written $\text{BoC}(\vec{c}, Q)$. Its radius is the **radius of convergence** of RhS(13a), and is written $\text{RoC}(\vec{c})$.^{♥4} So $\mathcal{R} := \text{RoC}(\vec{c})$ is always a value in $[0, +\infty]$, and $\text{BoC}(\vec{c}, Q) = \text{Bal}_{\mathcal{R}}(Q)$. \square

13b: RoC Lemma (Cauchy, 1821. Hadamard, 1888.) Contemplate power series $\text{PS}_{\vec{c}, Q}$, as in (13a). Let

$$\Omega := \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} \stackrel{\text{note}}{\in} [0, +\infty].$$

Then $\text{RoC}(\vec{c}) = 1/\Omega$ where, here, we interpret $\frac{1}{0}$ as $+\infty$ and $\frac{1}{+\infty}$ as 0. \diamond

^{♥4}The argument to RoC is a *sequence*. So we can write the RoC of PS $f(x) := \sum_{n=0}^\infty n^2 x^n$ as $\text{RoC}(n \mapsto n^2)$, but **not** as $\text{RoC}(n^2)$. nor as $\text{RoC}(f)$.

Proof sketch. Set $a_n := |c_n|$. Consider convergence at a non-negative $x \in \mathbb{R}$. Applying the Root-test,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n x^n|} &= \limsup_{n \rightarrow \infty} [x \cdot \sqrt[n]{a_n}] \\ &= x \cdot \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = x \cdot \Omega =: \Lambda. \end{aligned}$$

So Λ is less/greater than 1, as x is less/greater than $\frac{1}{\Omega}$. ♦

14: Three examples. ASIDE: On a set Ω , each subset $B \subset \Omega$ engenders $\mathbf{1}_B$, the “**indicator function** of B ”. It is the fnc $\Omega \rightarrow \{0, 1\}$ sending points in B to 1, and pts in its complement, $B^c := \Omega \setminus B$, to 0. [So $\mathbf{1}_B + \mathbf{1}_{B^c}$ is constant-1.] E.g, $\mathbf{1}_{\text{Primes}}(5)=1$ and $\mathbf{1}_{\text{Primes}}(9)=0$.

Let’s apply the above (13b). Define

$$\mathbb{P} := \text{Primes}; D := \text{Odds}; S := \{1 + n^2 \mid n \in \mathbb{N}\}.$$

Consider this power series:

$$14a: \sum_{n=0}^{\infty} 3^n \cdot \mathbf{1}_{\mathbb{P}}(n) \cdot x^n = 9x^2 + 27x^3 + 243x^5 + \dots$$

Its RoC is $1/3$, since there are ∞ many primes.

A funkier PS, centered at 8, is

$$14b: \sum_{k=0}^{\infty} \left[3^k \cdot \mathbf{1}_D(k) + 4^k \cdot \mathbf{1}_S(k) \right] \cdot [x - 8]^k.$$

Since $\sqrt[n]{3^n + 4^n} \xrightarrow{n} 4$, and $|S| = \infty$, the RoC is $\frac{1}{4}$.

Even more interesting is this PS:

$$14c: \sum_{n=0}^{\infty} \left[5^n \cdot \mathbf{1}_{\mathbb{P}}(n) \cdot \mathbf{1}_S(n) \right] \cdot x^n.$$

As of March2017, its RoC is unknown. If there are ∞ many primes^{♥5} of form $1 + n^2$ (conjectured, but unproven) then $\text{RoC} = \frac{1}{5}$; otherwise $\text{RoC} = \infty$, and the PS is a polynomial. □

^{♥5}For the curious, see Wikipedia on Landau’s problems.

15: Lemma. For each $K \in \mathbb{R}$: $\lim_{x \nearrow \infty} \sqrt[x]{x^K} = 1$.
Moreover, for each rational function $h() := \frac{p()}{q()}$ which is eventually positive, $\lim_{n \nearrow \infty} \sqrt[n]{h(n)} = 1$.

Proof. Use L’Hôpital’s rule. Etc. ♦

16: Same-RoC lemma. Consider a sequence $\vec{c} = (c_0, c_1, \dots) \subset \mathbb{C}$, and let $\mathcal{R} := \text{RoC}(\vec{c})$. For each natnum K , and for each rational function $g \neq \text{Zip}$, these coefficient sequences

$$i: (0, \dots, 0, c_K, c_{K+1}, c_{K+2}, \dots)$$

$$ii: (c_K, c_{K+1}, c_{K+2}, \dots)$$

$$iii: (g(n) \cdot c_n)_{n=0}^{\infty}$$

give rise to power-series with $\text{RoC} = \mathcal{R}$. ♦

Proof sketch. Parts (i) and (ii) follow from (13b). Part (iii) follows from (15) and (13b). ♦

17: Diff/Integrate a PS. We differentiate and integrate, term-by-term, the $G := \text{PS}_{\vec{c}, 0}$ power-series:

$$F(x) = \sum_{j=1}^{\infty} b_j \cdot x^j, \text{ where } b_j := \frac{1}{j} \cdot c_{j-1}.$$

$$17a: G(x) = \sum_{k=0}^{\infty} c_k \cdot x^k.$$

$$H(x) = \sum_{\ell=0}^{\infty} d_{\ell} \cdot x^{\ell}, \text{ where } d_{\ell} := [\ell+1] \cdot c_{\ell+1}.$$

Lemma (16) tells us that the three PSes have the same RoC.

Observe that $\text{PS}_{\vec{d}}$ is the term-by-term derivative of $\text{PS}_{\vec{c}}$. And $\text{PS}_{\vec{b}}$ is the term-by-term integral of $\text{PS}_{\vec{c}}$. Does the same relation hold between the *functions* that these PSes determine? □

17b: Term-by-term PS Theorem. Given a sequence $\vec{c} \subset \mathbb{R}$, define sequences/fncs $\vec{b}, \vec{d}, F, G, H$ by (17a) and let $\mathcal{R} := \text{RoC}(\vec{c})$. Then

$$\dagger: \text{RoC}(\vec{b}) = \mathcal{R} = \text{RoC}(\vec{d}).$$

With $B := \text{BoC}(\vec{c})$, moreover,

$$\ddagger: \forall z \in B: F(z) = \int_0^z G.$$

And G is in $C^{\infty}(B \rightarrow \mathbb{R})$, with $G' = H$. ♦

17c: Coro. Suppose PS $G(x) := \sum_{j=0}^{\infty} c_j \cdot [x - Q]^j$ has positive RoC. Then this PS is the Taylor series of G , centered at Q . \diamond

Pf of (17b). We'll establish that $G'=H$; the integral result (§) follows analogously. To fix a posreal $\rho < \mathcal{R}$, let $U := \text{Bal}_{\rho}(0)$, and prove $G'=H$ when restricted to U . We will apply the DUC Thm (Derivative uniform-convergence) from *notes-AdvCalc.pdf* to these fncs (defined only on U)

$$f_n(x) := \sum_{j \in [0..n]} c_j x^j.$$

By definition of coeff-sequence \vec{d} from (17a),

$$f'_n(x) = \sum_{k \in [0..n]} d_k x^k.$$

In order to show that $\text{seq } (f'_n)_{n=1}^{\infty}$ is sup-norm Cauchy, pick a number V with $\rho < V < \mathcal{R}$.

Now $\frac{1}{V} > \limsup_{n \rightarrow \infty} \sqrt[n]{|d_n|}$ since, by (16), $\text{RoC}(\vec{d})$ equals \mathcal{R} . Thus there is an index K with

$$\forall n \geq K: \sqrt[n]{|d_n|} < \frac{1}{V}.$$

We henceforth only consider indices n dominating K . For each $k \geq n$, then,

$$17d: \quad |d_k| \leq 1/V^k.$$

Sup-norm. For $x \in U$ and indices $\ell > n$,

$$f'_\ell(x) - f'_n(x) = \sum_{k \in [n.. \ell]} d_k x^k.$$

From (17d), then,

$$|f'_\ell(x) - f'_n(x)| \leq \sum_{k=n}^{\infty} \frac{|x|^k}{V^k}.$$

Since U owns x ,

$$|f'_\ell(x) - f'_n(x)| \leq \sum_{k=n}^{\infty} \frac{\rho^k}{V^k} = \left[\frac{\rho}{V}\right]^n \cdot C,$$

where C is the positive constant $1/[1 - \frac{\rho}{V}]$.

Taking a supremum over all $x \in U$ yields

$$17e: \quad \|f'_\ell - f'_n\| \leq \left[\frac{\rho}{V}\right]^n \cdot C,$$

for each pair $\ell > n \geq K$. Sending $n \nearrow \infty$ sends $\text{RhS}(17e) \rightarrow 0$.

The limit $\lim_n f_n(0)$ exists, equaling c_0 . Now apply the DUC Thm. \diamond

A power-series with a new center. We show that a function defined by a PS is analytic in its entire ball-of-convergence.

18: The setting. We have a point $P \in \mathbb{C}$ and a sequence $\vec{a} \subset \mathbb{C}$ such that $\alpha \in (0, +\infty]$, where $\alpha := \text{RoC}(\vec{a})$. This engenders a \mathbb{C}^∞ -fnc from $\text{Bal}_\alpha(P) \rightarrow \mathbb{C}$, by

$$18a: \quad \mathcal{F}(z) := \sum_{k=0}^{\infty} a_k \cdot [z - P]^k.$$

Fix a new center $Q \in \mathbb{C}$ with $|Q - P| < \alpha$. Thus

$$18b: \quad \beta \in (0, +\infty], \text{ where } \beta := \alpha - |Q - P|. \quad \square$$

Moreover, $\text{Bal}_\beta(Q) \subset \text{Bal}_\alpha(P)$.

19: New-center theorem. Take $P, Q, \alpha, \beta, \vec{a}$ and \vec{b} from (18). For each natnum k , this summation is absolutely convergent:

$$19a: \quad b_k := \sum_{N=k}^{\infty} a_N \cdot \binom{N}{k} \cdot Q^{N-k} \in \mathbb{C}.$$

Moreover, $\text{RoC}(\vec{b}) \geq \beta > 0$. This value

$$19b: \quad \mathcal{G}(z) := \sum_{k=0}^{\infty} b_k \cdot [z - Q]^k,$$

19c: agrees with $\mathcal{F}(z)$, for each $z \in \text{Bal}_\beta(Q)$.

Lastly, for each natnum k ,

$$19d: \quad b_k = \frac{1}{k!} \cdot \mathcal{F}^{(k)}(Q).$$

In other words, RhS(19b) is the Taylor series for \mathcal{F} , centered at Q . \diamond

Proof. WLOG $P = 0$. Fix a point $Z \in \text{Bal}_\beta(Q)$. Writing $Z = Q + [Z - Q]$, its N^{th} -power is

$$Z^N = \sum_{k=0}^N \binom{N}{k} \cdot Q^{N-k} \cdot [Z - Q]^k.$$

Thus, since $Z \in \text{Bal}_\alpha(P)$,

$$\begin{aligned} f(Z) &= \sum_{N=0}^{\infty} a_N \cdot Z^N \\ &= \sum_{N=0}^{\infty} \sum_{k=0}^N \underbrace{a_N \cdot \binom{N}{k} \cdot Q^{N-k}}_{h_{N,k}} \cdot [Z - Q]^k. \end{aligned}$$

This is a sum, in a certain order, over the set $H := \{(N, k) \in \mathbb{N} \times \mathbb{N} \mid N \geq k\}$. We need this sum to be absolutely convergent. The sum $\sum_{N=0}^{\infty} \sum_{k=0}^N |h_{N,k}|$ equals

$$*: \quad \sum_{N=0}^{\infty} \sum_{k=0}^N |a_N| \cdot \binom{N}{k} \cdot |Q|^{N-k} \cdot |Z - Q|^k = \sum_{N=0}^{\infty} |a_N| \cdot Y^N,$$

where $Y := |Q| + |Z - Q|$. From $Z \in \text{Bal}_\alpha(0)$ and (18b), we conclude that $Y < \alpha$. From the proof of Root-test lemma (12, P.6), the righthand side of (*) is finite.

Since $\mathbf{S} := \sum_{N=0}^{\infty} \sum_{k=0}^N |h_{N,k}|$ is finite, we can reverse the order of summation and conclude that

$$\begin{aligned} \mathbf{S} &= \sum_{k=0}^{\infty} \sum_{N=k}^{\infty} |h_{N,k}| \\ &= \sum_{k=0}^{\infty} \left[\sum_{N=k}^{\infty} |a_N| \cdot \binom{N}{k} \cdot |Q|^{N-k} \right] \cdot |Z - Q|^k. \end{aligned}$$

We could have chosen our $Z \neq Q$, thus allowing division by $|Z - Q|^k$. Hence, each bracketed sum is finite. So each sum in (19a) is absolutely convergent, and we have a well-defined number b_k .

For a general $Z \in \text{Bal}_\alpha(0)$, reversing the original sum gives

$$\begin{aligned} f(Z) &= \sum_{k=0}^{\infty} \sum_{N=k}^{\infty} h_{N,k} \\ &= \sum_{k=0}^{\infty} \left[\sum_{N=k}^{\infty} a_N \cdot \binom{N}{k} \cdot Q^{N-k} \right] \cdot [Z - Q]^k, \end{aligned}$$

which equals $\sum_{k=0}^{\infty} b_k \cdot [Z - Q]^k$.

Establishing (19d). Corollary 17c tells us that

$$k! \cdot b_k \stackrel{\text{by (17c)}}{=} \mathcal{G}^{(k)}(Q) \stackrel{\text{by (19c)}}{=} \mathcal{F}^{(k)}(Q). \quad \blacklozenge$$

20: Prop'n. Power-series

$$*: \quad \mathcal{F}(z) := \sum_{n=0}^{\infty} a_n \cdot [z - Q]^n$$

has positive RoC. Suppose \vec{y} is a sequence of distinct complex numbers converging to Q , such that

$$\forall j \in \mathbb{Z}_+: \quad \mathcal{F}(y_j) = 0.$$

Then \vec{a} is all-zero, and \mathcal{F} is the zero function. \diamond

Proof. WLOG, each $y_j \neq Q$. FTSOC, suppose $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$; let L be the smallest index with $a_L \neq 0$. Formally dividing (*) by $[z - Q]^L$ gives PS

$$\mathcal{G}(z) := \sum_{k=0}^{\infty} b_k \cdot [z - Q]^k,$$

where each $b_k := a_{L+k}$. Each $y_j - Q \neq 0$, so

$$\mathcal{G}(y_j) = \mathcal{F}(y_j) / [y_j - Q]^L = 0.$$

But $\text{RoC}(\vec{\mathbf{b}}) = \text{RoC}(\vec{\mathbf{a}}) > 0$, so \mathcal{G} is cts in a nbhd of Q , and thus $\mathcal{G}(Q) = \lim(\mathcal{G}(\vec{\mathbf{y}})) = 0$. This contradicts that $\mathcal{G}(Q) = b_0 = a_L \neq 0$. \blacklozenge

21: PS Uniqueness Thm. *Imagine power-series*

$$\begin{aligned} \mathcal{F}(z) &:= \sum_{n=0}^{\infty} a_n \cdot [z - P]^n \quad \text{and} \\ \mathcal{G}(z) &:= \sum_{n=0}^{\infty} b_n \cdot [z - P]^n \end{aligned}$$

where $B := \text{BoC}(\vec{\mathbf{a}}) \cap \text{BoC}(\vec{\mathbf{b}})$ is non-void. Suppose there is a set $Y \subset B$ st. $\mathcal{F}|_Y = \mathcal{G}|_Y$, and Y has a cluster point, Q_0 , in B . Then $\vec{\mathbf{a}} = \vec{\mathbf{b}}$, so $\mathcal{F} = \mathcal{G}$. \blacklozenge

Remark. It does not suffice for Y to have a cluster-point on the *boundary* of B : Distinct functions $\mathcal{F}(z) := \sin(\frac{1}{z-7})$ and $\mathcal{G} := -\mathcal{F}$ have Taylor series with $\text{RoC} = 7$. Yet

$$\mathcal{F}(y_k) = 0 = \mathcal{G}(y_k), \quad \text{for each posit } k,$$

where $y_k := 7 + \frac{1}{2\pi k}$. \square

Proof of (21). Subtracting PSes gives us a PS

$$f(z) := \sum_{n=0}^{\infty} c_n \cdot [z - P]^n$$

so that $f|_Y \equiv 0$, making $\vec{\mathbf{c}} \stackrel{?}{=} \vec{\mathbf{0}}$ our goal.

For each $q \in B := \text{BoC}(\vec{\mathbf{c}})$, let $U(q)$ denote the *largest* centered-at- q open ball that fits inside B . By the **New-center thm**, the Taylor-series for f , centered at q , converges to f on all of $U(q)$.

Pick a Y -cluster-point $Q_0 \in B$. By (20), f is identically zero on $U(Q_0)$.

On the line-segment running between Q_0 and P , we can pick a (finite) list of points

$$Q_0, Q_1, \dots, Q_{K-1}, Q_K := P,$$

such that each $Q_k \in U(Q_{k-1})$. Arguing inductively, since f is identically zero on $U(Q_{k-1})$, the the Taylor-series at Q_k has all-zero coeffs. This therefore holds at P . So $\vec{\mathbf{c}} = (0, 0, 0, \dots)$. \blacklozenge

22: Coro. *Suppose \mathcal{F} and \mathcal{G} are analytic functions on some connected open set $V \subset \mathbb{C}$. If*

$$\{z \in V \mid \mathcal{F}(z) = \mathcal{G}(z)\}$$

has a cluster point in V , then $\mathcal{F} = \mathcal{G}$. \blacklozenge

Abel's theorem

We state this for a PS centered at $Q \in \mathbb{C}$.

23: Abel's Theorem. Consider a power-series

$$f(z) := \sum_{n=0}^{\infty} c_n \cdot [z - Q]^n,$$

with RoC $\mathcal{R} \in (0, \infty)$, as well as a point z_0 on the circle-of-convergence, $|z_0 - Q| = \mathcal{R}$. If $f(z_0)$ is finite, then

$$\lim_{t \nearrow 1} f(t \cdot z_0) = f(z_0).$$

I.e, f is continuous along radial lines. ♦

Reduction. WLOG $Q = 0$. WLOG $\mathcal{R} = 1$ and $z_0 = 1$. So

23a:
$$f(x) = \sum_{n=0}^{\infty} c_n \cdot x^n$$

with RoC(\vec{c}) = 1, and $\sum_{n=0}^{\infty} c_n$ finite. Subtract a constant from c_0 so that, WLOG,

23b:
$$0 = f(1) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} c_n \stackrel{\text{def}}{=} \lim_{\ell \nearrow \infty} s_\ell,$$

where $s_\ell := \sum_{n \in [0.. \ell]} c_n$ is the ℓ^{th} **partial-sum**. Having fixed $\varepsilon > 0$, we will show

$$\forall x \in [\beta, 1): |f(x)| \leq 2\varepsilon,$$

for a posreal $\beta < 1$ that we will produce. □

Pf. Consider points $x \in J := [0, 1)$. Limit (23b) says we can take \mathbf{L} so large that

$$\forall \ell \geq \mathbf{L}: |s_\ell| \leq \varepsilon.$$

And $\sum_{\ell=\mathbf{L}}^{\infty} s_\ell x^\ell$ equals $x^\mathbf{L} \cdot \sum_{n=0}^{\infty} s_{n+\mathbf{L}} x^n$. So

$$\begin{aligned} \left| [1-x] \cdot \sum_{\ell=\mathbf{L}}^{\infty} s_\ell x^\ell \right| &\leq x^\mathbf{L} \cdot [1-x] \cdot \sum_{n=0}^{\infty} \varepsilon x^n \\ &= x^\mathbf{L} \cdot \varepsilon \leq \varepsilon, \end{aligned}$$

since $|x| \leq 1$. Let's summarize.

23c:
$$\forall x \in J: \left| [1-x] \cdot \sum_{\ell \in [\mathbf{L}.. \infty)} s_\ell x^\ell \right| \leq \varepsilon.$$

Defining β . Wanting $[1 - \beta] \cdot \sum_{\ell \in [0.. \mathbf{L})} |s_\ell| < \varepsilon$, simply fix a $\beta < 1$ sufficiently close to 1. Thus

23d:
$$\forall x \in [\beta, 1): \left| [1-x] \cdot \sum_{\ell \in [0.. \mathbf{L})} s_\ell x^\ell \right| \leq \varepsilon.$$

Absolute convergence. At a point $x \in J$, series (23a) and $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ are *absolutely convergent*. So we can multiply the series and regroup to conclude:

$$\begin{aligned} \frac{1}{1-x} \cdot f(x) &= \left[\sum_{j=0}^{\infty} 1 \cdot x^j \right] \cdot \left[\sum_{k=0}^{\infty} c_k x^k \right] \\ &= \sum_{\ell=0}^{\infty} \left[\sum_{j+k=\ell} 1 \cdot c_k \right] x^\ell \stackrel{\text{note}}{=} \sum_{\ell=0}^{\infty} s_\ell x^\ell. \end{aligned}$$

For each $x \in [\beta, 1)$, then, (23d) and (23c) give

$$|f(x)| = \left| [1-x] \cdot \sum_{\ell=0}^{\infty} s_\ell x^\ell \right| \leq \varepsilon + \varepsilon. \quad \blacklozenge$$