Taylor's theorem in several guises

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Preliminaries. A polynomial $p()$ is an "***N-topped polynomial***" if $\mathrm{Deg}(p) < N$. So $x^2 - 2x$ and $x + \sqrt{7}$ are 3-topped, but $x^3 + x - 5$ is not. The set of N-topped polynomials is an N-dimensional VS.[♡1]

Taylor polynomials

The setting is an open interval $J \subset \mathbb{R}$. Use $\mathsf{Diff}^N = \mathsf{Diff}^N(J{\to}\mathbb{R})$ for the set of N-times differentiable fncs. This is a superset of

$$\mathbf{C}^N \quad = \quad \mathbf{C}^N(J{\to}\mathbb{R})\,;$$

those, whose N^{th} derivative is *continuous*.

For posint N, a point $Q{\in}J$ and $f \in \mathsf{Diff}^{N-1}$, define

1a: "*the Nth* ***Taylor polynomial*** *of f, centered at Q*"

to be the *unique* N-topped polynomial $p()$ whose *zero*th through $[N{-}1]^{st}$ derivatives, at Q, agree with those of f. I.e, these N equations hold:

$$p(Q) = f(Q), \quad p'(Q) = f'(Q), \quad p''(Q) = f''(Q), \quad \ldots, \quad p^{(N-1)}(Q) = f^{(N-1)}(Q)\,.$$

One easily checks that $p(x)$ must be the RhS of (1b), below. That is,

1b: $$\mathbf{T}_N(x) \;=\; \mathbf{T}^f_{N,Q}(x) \;:=\; \sum_{k\in[0\,..\,N)} \tfrac{f^{(k)}(Q)}{k!}\cdot[x-Q]^k$$

is the Nth Taylor polynomial of f.

[♡1]Abbreviations: VS for *vector space*; FTC for *Fundamental Theorem of Calculus*; posint for *positive integer*.

Define the "the Nth ***remainder term*** of f", written $\mathbf{R}^f_{N,Q}$ or just $\mathbf{R}_N$, by

1c: $$f(x) \;=:\; \mathbf{T}_N(x) + \mathbf{R}_N(x)\,.$$

Properties of $\mathbf{T}_N$. Use $\mathbf{T}_{N,Q}[f]$ as a synonym for $\mathbf{T}^f_{N,Q}$, with the same convention for the $\mathbf{R}_N$ operator and friends. Often times either the fnc, f, or the center-point, Q, is implicit, and we drop it from the notation.

For a scalar α and $f,g \in \mathsf{Diff}^{N-1}$, note that

2: $$\begin{aligned}\mathbf{T}_N[\alpha\cdot f] &= \alpha\cdot\mathbf{T}_N[f] \qquad \text{and}\\ \mathbf{T}_N[f+g] &= \mathbf{T}_N[f] + \mathbf{T}_N[g]\,.\end{aligned}$$

I.e, $\mathbf{T}_N$ is a *linear operator* from Diff^{N-1} to the VS of polynomials. Also, for $f \in \mathsf{Diff}^N$,

3: $$\mathbf{T}_N[f'] \;=\; \Big[\mathbf{T}_N[f]\Big]'\,.$$

That is, $\mathbf{T}_N$ commutes with differentiation; in symbols $\mathbf{T}_N \leftrightarrows \mathbf{D}$. Letting $\mathbf{I}$ denote the identity operator, we have that $\mathbf{I} = \mathbf{T}_N + \mathbf{R}_N$. Since $\mathbf{I}$ is linear and commutes with $\mathbf{D}$, so is/does $\mathbf{R}_N$.

Courtesy FTC, integrating (3) gives

$$\begin{aligned}\int_Q^x \mathbf{T}_{N,Q}[f'](t)\cdot\mathrm{d}t \;&=\; \mathbf{T}_{N,Q}[f](x) - \mathbf{T}_{N,Q}[f](Q)\\ &=\; \mathbf{T}^f_{N,Q}(x) - f(Q)\,.\end{aligned}$$

There is a corresponding statement about the remainder term. □

Taylor's Theorem for $\mathbb{R}$

We produce two estimates[♡2] of the remainder term. In the results below, $N \in \mathbb{Z}_+$.

4: Lem. *Fix an $f \in \mathsf{Diff}^N(J{\to}\mathbb{R})$. At each $x \in J$, function $q \mapsto \mathbf{T}^f_{N,q}(x)$ is differentiable, with value*

4′: $$\tfrac{\mathrm{d}}{\mathrm{d}q}\mathbf{T}^f_{N,q}(x) \quad = \quad f^{(N)}(q)\cdot\tfrac{[x-q]^{N-1}}{[N-1]!}\,.$$

And $\tfrac{\mathrm{d}}{\mathrm{d}q}\mathbf{R}^f_{N,q}(x) \;=\; -\tfrac{\mathrm{d}}{\mathrm{d}q}\mathbf{T}^f_{N,q}(x)$. ◊

[♡2]The TayThm-1 estimate, (5a), is called *Lagrange's form* of the Nth remainder term.

Our TayThm-2, (6a), is one of the many *Integral forms* of the remainder.

Proof. WLOG $x{=}7$. For natnum k and posint ℓ, define

$$A_k(q) \;:=\; \tfrac{f^{(k)}(q)}{k!}\cdot[7-q]^k\,, \quad \text{and}$$
$$B_\ell(q) \;:=\; \tfrac{f^{(\ell)}(q)}{[\ell-1]!}\cdot[7-q]^{\ell-1}\,.$$

Now $A_0(q) = f(q){\cdot}1$, so $\frac{\mathrm{d}}{\mathrm{d}q}A_0(q) = f'(q) = B_1(q)$. And for k positive, $\frac{\mathrm{d}}{\mathrm{d}q}A_k(q)$ equals

$$\tfrac{1}{k!}\Big[f^{(k+1)}(q){\cdot}[7-q]^k + f^{(k)}(q){\cdot}k{\cdot}[7-q]^{k-1}{\cdot}\,[-1]\Big]$$
$$\overset{\text{note}}{=\!=}\; B_{k+1}(q) \;-\; B_k(q)\,.$$

Thus $\frac{\mathrm{d}}{\mathrm{d}q}$ of $\mathbf{T}^f_{N,q}(7)$ is (notationally dropping the "q")

$$[B_N - B_{N-1}] + [B_{N-1} - B_{N-2}] + \cdots + [B_2 - B_1] + B_1\,.$$

Telescoping, this equals B_N, which is RhS($4'$). ♦

5: TayThm-1. *Consider an $f \in \mathsf{Diff}^N(J{\to}\mathbb{R})$. For distinct points $Q,x \in J$, there exists a point $c{=}c_{N,Q,x,f}$ strictly between Q and x, such that*

5a: $$\mathbf{R}^f_{N,Q}(x) \;=\; \tfrac{1}{N!}f^{(N)}(c)\cdot[x-Q]^N\,.$$

I.e, $$f(x) \;=\; \left[\sum_{k=0}^{N-1} f^{(k)}(Q){\cdot}\tfrac{[x-Q]^k}{k!}\right] + f^{(N)}(c){\cdot}\tfrac{[x-Q]^N}{N!}\,.\diamondsuit$$

Pf. WLOG $Q = 3$ and $x = 7$. Define $M \in \mathbb{R}$ by

5b: $$\mathbf{R}_{N,3}(7) \;=\; M\cdot\tfrac{[7-3]^N}{N!}\,.$$

(Possible, since $[7-3]^N \neq 0$.) Create a function

5c: $$\varphi(q) \;:=\; \mathbf{R}_{N,q}(7) - M\cdot\tfrac{[7-q]^N}{N!}\,.$$

Our "multiplier" M was defined so that $\varphi(3) = 0$. And $\varphi(7) = 0 - M{\cdot}0 = 0$. (We used that $[7-7]^N{=}0^N$ is zero, since N is positive.) (Exer.: Why is $\mathbf{R}_{N,7}(7) = 0$?)

Rolle's thm now applies to φ on $[3,7]$, asserting a point $c \in (3,7)$ st. $\varphi'(c) = 0$. By Lemma 4,

$$0 \;=\; -f^{(N)}(c)\cdot\tfrac{[7-c]^{N-1}}{[N-1]!} - M{\cdot}[-1]{\cdot}\tfrac{[7-c]^{N-1}}{[N-1]!}\,.$$

Since $c \neq 7$, quantity $[7-c]^{N-1}$ is not zero; so we may divide, to conclude that $0 = -f^{(N)}(c) + M$. ♦

6: TayThm-2. *For $f \in \mathbf{C}^N(J)$ and points $Q,x \in J$:*

6a: $$\mathbf{R}^f_{N,Q}(x) \;=\; \tfrac{1}{[N-1]!}\int_Q^x f^{(N)}(t)\cdot[x-t]^{N-1}\,\mathrm{d}t\,.\diamondsuit$$

Proof. WLOG $Q = 3$ and $x = 7$. Define this function

$$\theta(q) \;:=\; \mathbf{T}^f_{N,q}(7)\,.$$

This $\theta \in \mathbf{C}^1$, since $f \in \mathbf{C}^N$. Since $\theta(7) = f(7)$,

$$\mathbf{R}^f_{N,3}(7) \;\overset{\text{def}}{=\!=}\; f(7) - \mathbf{T}^f_{N,3}(7)$$
$$*: \qquad = \theta(7) - \theta(3) \;=\; \int_3^7 \theta'$$

by FTC, since θ' is continuous on J. And RhS($*$) equals RhS($6a$), courtesy Lemma 4. ♦

Aside: An incorrect formula. The 1st edition of Buck's text has (6a), but the the 3rd ed. (P.148, top) asserts that $\mathbf{R}^f_{N,Q}(x)$ equals

6b: $$W^f_{N,Q}(x) \;:=\; \tfrac{1}{[N-1]!}\int_Q^x f^{(N)}(t)\cdot[t-Q]^{N-1}\,\mathrm{d}t\,.$$

Just because RhS($6b$) differs from RhS($6a$) doesn't mean that it is wrong; there are several formulae for $\mathbf{R}_N$ ITOf integrals. The following example, however, indeed shows that $W^f_{N,Q} \neq \mathbf{R}^f_{N,Q}$.

Let $f := [z \mapsto z^3]$, $Q := 0$ and $N := 2$. Thus $\mathbf{T}_2(x)$ equals $0 + 0{\cdot}x$, so $\boxed{\mathbf{R}_2(x) = x^3}$. Since $f''(t) = 3{\cdot}2{\cdot}t$, our $W_2(x)$ equals

$$**: \quad \tfrac{1}{1!}\int_0^x 3{\cdot}2{\cdot}t\cdot[t-0]\cdot\,\mathrm{d}t \;=\; 2t^3\big|_{t=0}^{t=x} \;=\; 2x^3\,.$$

So $W_2(1) = 2 \neq 1 = \mathbf{R}_2(1)$. In contrast, RhS($6a$) equals $\int_0^x 3{\cdot}2{\cdot}t\cdot[x-t]\cdot\mathrm{d}t$, which equals

$$\left[x\cdot\int_0^x 3{\cdot}2{\cdot}t\cdot\mathrm{d}t\right] - \text{RhS}(**) \;=\; x\cdot 3x^2 - 2x^3\,.$$

And this, happily, equals $\mathbf{R}_2(x)$. □

Applications of Taylor's theorem

Here are five uses.

7: *Translating a polynomial.* Suppose you wish to express polynomial

$$p(x) \;=\; C_0 + C_1 x + \cdots + C_{N-1}x^{N-1}$$

in form $\sum_{k\in[0\,..\,N)} B_k{\cdot}[x-6]^k$. You *could* solve a system of N many linear eqns in N unknowns.

But p **equals** its Nth TayPoly (centered anywhere we want). So each B_k is the kth coeff of $\mathbf{T}^p_{N,6}$. Thus B_k equals $p^{(k)}(6)/k!$. □

8: Limits. *Suppose $f \in \mathsf{Diff}^2(\mathbb{R}{\to}\mathbb{R})$ and $\lim_{x\to\infty} f(x)$ exists in $\mathbb{R}$, and $\limsup_{x\to\infty}|f''(x)| < \infty$. Then $f'(x) \to 0$ as $x{\nearrow}\infty$.* ◊

Set-up for both proofs. WLOG $\boxed{*:\ \lim_{x\to\infty} f(x) = 0}$ and $|f''()| \leqslant 8$. □

Pf. Fixing $\varepsilon{>}0$, we will produce an $L{\in}\mathbb{R}$ for which:

$$\forall x \geqslant L: \quad |f'(x)| \leqslant 5\varepsilon\,.$$

Consider TayPoly $\mathbf{T}^f_{2,x}(y) = f(x) + f'(x)[y-x]$ for $N{=}2$. The TayThm-1 gives that for each pair $x{<}y$ there is a point $c_{x,y} \in (x,y)$ for which

$$f(y) \;=\; f(x) + f'(x)[y-x] \;+\; \tfrac{f''(c_{x,y})}{2}[y-x]^2.$$

Solve for $f'(x)$. Since $|f''(c_{x,y})| \leqslant 8$,

$$|f'(x)| \;\leqslant\; \left|\tfrac{f(y)-f(x)}{y-x}\right| + \tfrac{8}{2}{\cdot}[y-x]\,.$$

By $(*)$, there is $L \in \mathbb{R}$ st. $|f| \leqslant \varepsilon^2/2$ on $[L,\infty)$. Setting $y = x+\varepsilon$, we have that for each $x \geqslant L$:

$$\begin{aligned}|f'(x)| \;&\leqslant\; \left|\tfrac{\varepsilon^2}{y-x}\right| + 4{\cdot}[y-x] \\ &\leqslant\; \tfrac{\varepsilon^2}{\varepsilon} \;+\; 4\varepsilon \;=\; 5\varepsilon\,.\end{aligned}$$ ♦

Proof, barehands. Could $\beta := \liminf_{x\to\infty} f'(x) > 0$? No; for then the graph of f would grow at some minimum rate, thus could not have a horizontal asymptote. Hence $\beta \leqslant 0$. Similarly $\alpha \geqslant 0$, where $\alpha := \limsup_{x\to\infty} f'(x)$. The upshot: $\alpha \geqslant 0 \geqslant \beta$.

FTSOC, suppose $\alpha > \beta$. WLOG $\alpha > 0$; otherwise, replace f by $-f$. Set $\varepsilon := \alpha/3$.

Let a_1 be, say, the smallest non-negative "x-value" such that $f(a_1) \geqslant 2\varepsilon$. Since

$$\alpha \;>\; 2\varepsilon \;>\; \varepsilon \;>\; \beta\,,$$

the following process never stops: Take b_n to be the smallest "x" exceeding a_n for which $f(b_n) \leqslant \varepsilon$. Take a_{n+1} to be the smallest value exceeding b_n for which $f(a_{n+1}) \geqslant 2\varepsilon$.

Let $J_n := [a_n, b_n]$. Each restriction $f'\lfloor_{J_n} \geqslant \varepsilon$, for $n = 1, 2, \ldots$. By FTC,

$$\int_{b_n}^{a_n} f'' \;=\; f'(a_n) - f'(b_n) \;\geqslant\; 2\varepsilon - \varepsilon \;=\; \varepsilon\,.$$

So $\varepsilon \leqslant \int_{J_n}|f''| \leqslant 8{\cdot}[b_n - a_n]$. Hence each $b_n - a_n$ dominates $\frac{8}{\varepsilon}$; thus the numbers $a_n, b_n \nearrow \infty$.

By $(*)$, then, $f(b_n) - f(a_n) \to 0$, as $n{\nearrow}\infty$. But

$$\begin{aligned}f(b_n){-}f(a_n) \;&=\; \int_{a_n}^{b_n} f' \;\geqslant\; \int_{a_n}^{b_n} \varepsilon \\ &\geqslant\; [b_n - a_n]\varepsilon \;\geqslant\; \tfrac{8}{\varepsilon}{\cdot}\varepsilon \;=\; 8.\end{aligned}$$ ♦

N^{th}-derivative test for extrema. With $J \subset \mathbb{R}$ an open interval and $f \in \mathbf{C}^\infty(J{\to}\mathbb{R})$, suppose $Q{\in}J$ is a ***critical point*** for f, ie. $f'(Q) = 0$. We seek a test determining if f has a *strict local-max/min* at Q. To this end, define $\mathrm{MinNZD}(f,Q)$ to be

$$\text{9a:}\quad \inf\left\{k \in \mathbb{Z}_+ \;\middle|\; f^{(k)}(Q) \neq 0\right\} \overset{\text{note}}{\in} [1\,..\,\infty]\,.$$ □

9b: Min/Max Prop'n. With J, f, Q as above: *Suppose $N := \mathrm{MinNZD}(f,Q)$ is in $[2\,..\,\infty)$. When N is...*

...even: If $f^{(N)}(Q) > 0$ then f has a strict-local-min; else, f has a strict-local-max, at Q.

...odd: *Then f has a strict-SignChange at Q.* ◊

Pf. Let $V := f^{(N)}(Q)$, which is not zero. Since $f^{(N)}$ is continuous at Q, there is a small open interval $I \ni Q$ for which $f^{(N)}\rfloor_I$ has the same sign as V. Our Prop'n will be established by (9c), below.

Let $\mathbf{T} := \mathbf{T}^f_{N,Q}$ and $\mathbf{R} := \mathbf{R}^f_{N,Q}$. Recall that $f^{(k)}(Q) = 0$, for each $k \in [1\,..\,N)$. For each x, then, $\mathbf{T}(x) = f(Q)$, and thus $\mathbf{R}(x) = f(x) - f(Q)$.

For $x \neq Q$, there is a point τ_x, between x and Q, with $\mathbf{R}(x) = \frac{1}{N!} f^{(N)}(\tau_x) \cdot [x - Q]^N$. Taking $x \in I$ forces $\tau_x \in I$, so

$$f^{(N)}(\tau_x) \text{ has the same sign as } V.$$

But $f(x) - f(Q) = \mathbf{R}(x)$, so the sign-fnc gives

9c: $$\mathrm{Sgn}\big(f(x) - f(Q)\big) = \mathrm{Sgn}(V) \cdot [\mathrm{Sgn}(x - Q)]^N, \quad \text{for each } x \in I \text{ with } x \neq Q.$$ ♦

10: Lemma. *Set $J := [-1, 1]$. Imagine a function $f \in \mathsf{Diff}^3(J \to \mathbb{R})$ with*

10a: $$f(0) = 0 = f'(0).$$

10b: $$f(-1) = 0 \quad \text{and} \quad f(1) = 1.$$

Then there exists $\tau \in J^\circ$ with $f'''(\tau) \geqslant 3$. ◊

Counting degrees-of-freedom. What would it mean if each derivative $\{f^{(k)}\}_{k=3}^\infty$ were identically-zero? Then f is 3-topped polynomial (i.e, at most quadratic) and so comes from a 3-dim'al VS.

But (10a,10b) are *four* conditions, making it plausible that we cannot fulfill them when constrained to the VS of 3-topped polynomials. □

Proof. ISTProduce points $\alpha, \beta \in J^\circ$ with

10c: $$f^{(3)}(\alpha) + f^{(3)}(\beta) \geqslant 6.$$

Let $\mathbf{T} := \mathbf{T}^f_{3,0}$ and $\mathbf{R} := \mathbf{R}^f_{3,0}$. Define a number M by

$$\mathbf{T}(x) = f(0) + f'(0)x + \overbrace{\tfrac{f'(0)}{2}}^{M} x^2 \overset{\text{by (10a)}}{=\!=\!=} M \cdot x^2.$$

Thus $\mathbf{T}(\pm 1) = M \cdot [\pm 1]^2 = M$. From (10b), then,

$$1 = f(1) - f(-1) = [M + \mathbf{R}(1)] - [M + \mathbf{R}(-1)] = \mathbf{R}(1) - \mathbf{R}(-1).$$

By TayThm-1, $\exists \alpha \in (-1, 0)$ and $\exists \beta \in (0, 1)$ with

$$\mathbf{R}(1) = \tfrac{1}{6} \cdot f^{(3)}(\beta) \cdot [+1 - 0]^3 = \tfrac{1}{6} \cdot f^{(3)}(\beta);$$
$$\mathbf{R}(-1) = \tfrac{1}{6} \cdot f^{(3)}(\alpha) \cdot [-1 - 0]^3 = -\tfrac{1}{6} \cdot f^{(3)}(\alpha).$$

This, using the previous display, yields (10c). ♦

DiffyQ. A fnc $f \in \mathsf{Diff}^2(\mathbb{R} \to \mathbb{R})$ with $f(7) = 0$ and $f'(7) = 0$, satisfies

†: $$f'' + f = \mathbf{0}. \quad \text{[I.e, the zero-fnc.]}$$

Prove that $f = \mathbf{0}$, using Taylor's thm, as follows:

First, show that $f \in \mathbf{C}^\infty$. Then, for a *fixed* $x_0 \in \mathbb{R}$, argue that $\big|f(x_0)\big|$ is as small as desired, by upper-bounding with Taylor-remainder terms from **Our Taylor's pamphlet** on the Teaching Page.

Proof. Let $\|\cdot\|$ mean $\|\cdot\|_{\sup}$. Induction on $n \in \mathbb{N}$ shows that each $f^{(n)}$ is diff'able. Moreover

‡: $$\forall k, \ell \in \mathbb{N}: \quad \text{If } k \equiv_4 \ell \text{ then } f^{(k)} = f^{(\ell)}.$$

Since $f^{(2)}(7) = -f(7) = 0$ and $f^{(3)}(7) = -f'(7) = 0$, property (‡) tells us that each $\mathbf{T}^f_{n,7}$ is the zero-function. In particular, with $\mathbf{R}_n$ denoting $\mathbf{R}^f_{n,7}$,

$$\forall n \in \mathbb{Z}_+: \quad \mathbf{R}_n(x_0) = f(x_0).$$

Upper bounds. Let J be the compact interval going from 7 to x_0. Let $M_n := \|f^{(n)}\rfloor_J\|$. Define

$$\mathbf{M} := \mathrm{Max}(M_0, M_1, M_2, M_3).$$

Then $\mathbf{M}$ dominates each $\|f^{(n)}\rfloor_J\|$, courtesy (‡). From TayThm-2, then, $|f(x_0)|$ equals $|\mathbf{R}_n(x_0)|$ which equals

$$\begin{aligned}
&\tfrac{1}{[n-1]!}\cdot\left|\int_7^{x_0} f^{(n)}(t)\cdot[x_0-t]^{n-1}\cdot\mathrm{d}t\right| \\
&\leqslant \tfrac{1}{[n-1]!}\int_7^{x_0}|f^{(n)}(t)|\cdot|x_0-t|^{n-1}\cdot\mathrm{d}t \\
&\leqslant \tfrac{\mathbf{M}}{[n-1]!}\int_7^{x_0}|x_0-t|^{n-1}\cdot\mathrm{d}t\,.
\end{aligned}$$

Let $L := \mathrm{Len}(J)$. Then

$$|f(x_0)| \;\leqslant\; \tfrac{\mathbf{M}}{[n-1]!}\int_7^{x_0} L^{n-1}\cdot\mathrm{d}t \;=\; \mathbf{M}\cdot\tfrac{L^n}{[n-1]!}\,.$$

This last goes to zero, as $n \nearrow \infty$. ♦

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Radius of Convergence

Series notations. Customs about how "series" is used in the context of "convergence of a series" are a bit strange. A "***series*** $\vec{\mathbf{e}}$" is a *sequence* $\vec{\mathbf{e}} = (e_k)_{k=0}^{\infty}$, but[♡3] where the word "series" hints to the reader our interest in its *sum* $\sum(\vec{\mathbf{e}})$. This sum is the limit –when it exists– of the corresponding "partial-sum sequence" $\vec{\mathbf{s}}$, where

11: $$s_N \;:=\; \sum_{k\in[0\,..\,N)} e_k$$

Use $\boxed{\vec{\mathbf{s}} = \mathsf{P}\Sigma(\vec{\mathbf{e}})}$ to indicate this partial-sum relation between sequences. Phrase "series $\vec{\mathbf{e}}$ is convergent" means that $\lim(\vec{\mathbf{s}})$ exists and is finite. So $\sum(\vec{\mathbf{e}}) := \lim(\vec{\mathbf{s}})$.

To clarify, the nth partial sum means the sum of the first n terms, regardless of the initial index. For example, suppose $\vec{\mathbf{b}} = (b_\ell)_{\ell=5}^{\infty}$, and $\vec{\mathbf{e}} = \mathsf{P}\Sigma(\vec{\mathbf{b}})$. Then $e_3 = b_5+b_6+b_7$, and $e_0 = 0$.

Example: Let $\vec{\mathbf{b}} := (k^2)_{k=1}^{\infty}$ and $\vec{\mathbf{a}} := \mathsf{P}\Sigma(\vec{\mathbf{b}})$. Then $a_n = \frac{1}{6}\cdot[2n^3+3n^2+n]$. □

12: Root-test lemma. *Given a series $\vec{\mathbf{e}} \subset \mathbb{C}$, define*

$$*:\quad \Lambda \;:=\; \limsup_{n\to\infty}\sqrt[n]{|e_n|} \;\overset{note}{\in}\; [0,+\infty]\,.$$

If $\Lambda < 1$ then $\vec{\mathbf{e}}$ is an absolutely-convergent series.

If $\Lambda > 1$ then $\vec{\mathbf{e}}$ is "magnificently divergent" Not only $|e_n| \not\to 0$, but indeed $\limsup_{n\to\infty}|e_n| = +\infty$. ♢

Proof. Let $a_n := |e_n|$,

$\boxed{\text{Case: When } \Lambda < 1.}$ ISTShow that $\vec{\mathbf{a}}$ is a convergent series. Pick ρ with $\Lambda < \rho < 1$. Take K large enough that $\sup_{n\geqslant K}\sqrt[n]{a_n} \leqslant \rho$. Hence $\sum_{n\geqslant K} a_n \leqslant \sum_{n\geqslant K}\rho^n < \infty$. And $\sum_{n\in[1\,..\,K]} a_n < \infty$.

$\boxed{\text{Case: When } \Lambda > 1.}$ Pick ρ with $1 < \rho < \Lambda$. By $(*)$, the set $S := \{n \mid \sqrt[n]{a_n} > \rho\}$ is infinite. And each $n \in S$ has $a_n > \rho^n$. ♦

A function $f\colon\mathbb{R}\to\mathbb{R}$ is ***eventually positive*** if $\boxed{\exists K \text{ s.t } \forall x \geqslant K\colon f(x) > 0}$. Thus a degree-$k$ poly,

$$f(x) \;:=\; C_k x^k + \cdots + C_1 x + C_0\,,$$

IFF it has positive leading-coeff, $C_k > 0$.

Power-series notation. A sequence $\vec{\mathbf{c}} \subset \mathbb{C}$ and point $Q \in \mathbb{C}$ determine a ***power series***

13a: $$\mathrm{PS}_{\vec{\mathbf{c}},Q}(z) \;:=\; \sum_{n=0}^{\infty} c_n\cdot[z-Q]^n\,.$$

From the notation we sometimes drop the the center of expansion, just writing $\mathrm{PS}_{\vec{\mathbf{c}}}$. This is especially true when the center of expansion is $0\in\mathbb{C}$.

Use "PS" to abbreviate the phrase "power series". □

Radius of Convergence. The set of $z \in \mathbb{C}$ for which RhS(13a) converges is called the "***set-of-convergence***". We write it $\mathsf{SoC}(\vec{\mathbf{c}}, Q)$

It will turn out that the SoC comprises an open ball, possibly of radius 0 or ∞, together with some of the points on the boundary of this ball. This open ***ball of convergence*** is written $\mathsf{BoC}(\vec{\mathbf{c}}, Q)$. Its radius is the ***radius of convergence*** of RhS(13a), and is written $\mathsf{RoC}(\vec{\mathbf{c}})$.[♡4] So $\mathcal{R} := \mathsf{RoC}(\vec{\mathbf{c}})$ is always a value in $[0,+\infty]$, and $\mathsf{BoC}(\vec{\mathbf{c}}, Q) = \mathrm{Bal}_{\mathcal{R}}(Q)$. □

[♡3] The index will usually start at zero, but it doesn't have to. The sequence $\vec{\mathbf{e}}$ might be $(e_k)_{k=24}^{\infty}$, or $(e_k)_{k=-5}^{\infty}$.

[♡4] The argument to RoC is a *sequence*. So we can write the RoC of PS $f(x) := \sum_{n=0}^{\infty} n^2 x^n$ as $\mathsf{RoC}(n \mapsto n^2)$, but **not** as $\mathsf{RoC}(n^2)$. nor as $\mathsf{RoC}(f)$.

13b: RoC Lemma (Cauchy, 1821. Hadamard, 1888.) *Contemplate power series* $\mathrm{PS}_{\vec{\mathbf{c}},Q}$, *as in* (13a). *Let*

$$S \coloneqq \limsup_{n\to\infty} \sqrt[n]{|c_n|} \overset{note}{\in} [0,+\infty].$$

Then $\mathsf{RoC}(\vec{\mathbf{c}}) = 1/S$ *where, here, we interpret* $\frac{1}{0}$ *as* $+\infty$ *and* $\frac{1}{+\infty}$ *as* 0. ◊

Proof sketch. Set $a_n \coloneqq |c_n|$. ISTConsider convergence at a non-negative $x\in\mathbb{R}$. Then

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{|c_n x^n|} &= \limsup_{n\to\infty} [x \cdot \sqrt[n]{a_n}] \\ &= x \cdot \limsup_{n\to\infty} \sqrt[n]{a_n} = x \cdot S =: \Lambda .\end{aligned}$$

So Λ is less/greater than 1, as x is less/greater than $\frac{1}{S}$. ♦

14: *Three examples.* Let's apply the above (13b). Define

$$\mathbb{P} \coloneqq \text{Primes};\ D \coloneqq \text{Odds};\ E \coloneqq \{1+n^2 \mid n\in\mathbb{N}\}.$$

Consider this power series:

$$\text{14a:}\quad \sum_{n=0}^{\infty} 3^n\cdot\mathbf{1}_{\mathbb{P}}(n)\cdot x^n = 9x^2+27x^3+243x^5+\dots.$$

Its RoC is 1/3.

A funkier PS, centered at 8, is

$$\text{14b:}\quad \sum_{k=0}^{\infty} [3^k\cdot\mathbf{1}_D(k)+4^k\cdot\mathbf{1}_E(k)]\cdot[x-8]^k$$

Since $\sqrt[n]{3^n+4^n} \xrightarrow{n} 4$ and $|E|=\infty$, the RoC is $\frac{1}{4}$.

Even more interesting is this PS:

$$\text{14c:}\quad \sum_{n=0}^{\infty} [5^n\cdot\mathbf{1}_{\mathbb{P}}(n)\cdot\mathbf{1}_E(n)]\cdot x^n .$$

As of April2010, its RoC is unknown. If there are ∞ly many primes of form $1+n^2$ (conjectured, but unproven) then $\mathsf{RoC}=\frac{1}{5}$; otherwise $\mathsf{RoC}=\infty$. □

15: Lemma. *For each* $K\in\mathbb{R}$: $\lim_{x\nearrow\infty} \sqrt[x]{x^K} = 1$.

Moreoever, for each rational function $h() \coloneqq \frac{p()}{q()}$ *which is* <u>*eventually*</u> <u>*positive*</u>, $\lim_{n\nearrow\infty} \sqrt[n]{h(n)} = 1$.

Proof. Use L'Hôpital's rule. Etc. ◊

16: Same-RoC lemma. *Consider a sequence* $\vec{\mathbf{c}} = (c_0, c_1, \dots) \subset \mathbb{C}$, *and let* $\mathcal{R} \coloneqq \mathsf{RoC}(\vec{\mathbf{c}})$. *For each natnum* K, *and for each eventually-positive rational function* h, *these coefficient sequences*

i: $(0, \overset{K}{\dots}, 0, c_K, c_{K+1}, c_{K+2}, \dots)$

ii: $(c_K, c_{K+1}, c_{K+2}, \dots)$

iii: $\big(h(n)\cdot c_n\big)_{n=0}^{\infty}$

give rise to power-series with $\mathsf{RoC}=\mathcal{R}$. ◊

Proof sketch. Parts (i) and (ii) follow from (13b). Part (iii) follows from (15) and (13b). ♦

17: *Diff/Integrate a PS.* We differentiate and integrate, term-by-term, the $G \coloneqq \mathrm{PS}_{\vec{\mathbf{c}},0}$ power-series:

$$\begin{aligned} F(x) &= \textstyle\sum_{j=1}^{\infty} b_j\cdot x^j, \text{ where } b_j \coloneqq \tfrac{1}{j}\cdot c_{j-1}. \\ \text{17a:}\quad G(x) &= \textstyle\sum_{k=0}^{\infty} c_k\cdot x^k . \\ H(x) &= \textstyle\sum_{\ell=0}^{\infty} d_\ell\cdot x^\ell, \text{ where } d_\ell \coloneqq [\ell+1]\cdot c_{\ell+1}.\end{aligned}$$

Lemma (16) tells us that the three PSes have the same RoC.

Observe that $\mathrm{PS}_{\vec{\mathbf{d}}}$ is the term-by-term derivative of $\mathrm{PS}_{\vec{\mathbf{c}}}$. And $\mathrm{PS}_{\vec{\mathbf{b}}}$ is the term-by-term integral of $\mathrm{PS}_{\vec{\mathbf{c}}}$. Does the same relation hold between the *functions* that these PSes determine? □

17b: Term-by-term PS Theorem. *Given a sequence* $\vec{\mathbf{c}}\subset\mathbb{R}$, *define sequences/fncs* $\vec{\mathbf{b}}, \vec{\mathbf{d}}, F, G, H$ *by* (17a) *and let* $\mathcal{R} \coloneqq \mathsf{RoC}(\vec{\mathbf{c}})$. *Then*

$$\dagger:\qquad \mathsf{RoC}(\vec{\mathbf{b}}) = \mathcal{R} = \mathsf{RoC}(\vec{\mathbf{d}}).$$

With $B \coloneqq \mathsf{BoC}(\vec{\mathbf{c}})$, *moreover,*

$$\ddagger:\qquad \forall z\in B:\quad F(z) = \int_0^z G .$$

And G *is in* $\mathbf{C}^\infty(B\to\mathbb{R})$, *with* $G' = H$. ◊

17c: Coro. *Suppose PS $G(x) := \sum_{j=0}^{\infty} c_j \cdot [x-Q]^j$ has positive* RoC. *Then this PS is the Taylor series of G, centered at Q.* ◊

Pf of (17b). We'll establish that $G'=H$; the integral result (‡) follows analogously. ISTo fix a posreal $\rho < \mathcal{R}$, let $U := \mathrm{Bal}_\rho(0)$, and prove $G'=H$ when *restricted to* U. We will apply the DUC Thm (Derivative uniform-convergence) from *notes-AdvCalc.pdf* to these fncs (defined *only* on U)

$$f_n(x) \quad := \quad \sum_{j\in[0\,..\,n]} c_j x^j \,.$$

By definition of coeff-sequence $\vec{\mathbf{d}}$ from (17a),

$$f_n'(x) \quad = \quad \sum_{k\in[0\,..\,n)} d_k x^k \,.$$

In order to show that seq $(f_n')_{n=1}^{\infty}$ is sup-norm Cauchy, pick a number V with $\rho < V < \mathcal{R}$.

Now $\frac{1}{V} > \limsup_{n\to\infty} \sqrt[n]{|d_n|}$ since, by (16), RoC($\vec{\mathbf{d}}$) equals $\mathcal{R}$. Thus there is an index K with

$$\forall n \geqslant K: \quad \sqrt[n]{|d_n|} < \tfrac{1}{V} \,.$$

We henceforth only consider indices n dominating K. For each $k \geqslant n$, then,

17d: $$|d_k| \quad \leqslant \quad 1/V^k \,.$$

Sup-norm. For $x \in U$ and indices $\ell > n$,

$$f_\ell'(x) - f_n'(x) \;=\; \sum_{k\in[n\,..\,\ell)} d_k x^k \,.$$

From (17d), then,

$$\left|f_\ell'(x) - f_n'(x)\right| \;\leqslant\; \sum_{k=n}^{\infty} \frac{|x|^k}{V^k} \,.$$

Since U owns x,

$$\left|f_\ell'(x) - f_n'(x)\right| \;\leqslant\; \sum_{k=n}^{\infty} \frac{\rho^k}{V^k} \;=\; \left[\frac{\rho}{V}\right]^n \cdot C \,,$$

where C is the positive constant $1/[1-\frac{\rho}{V}]$.

Taking a supremum over all $x \in U$ yields

17e: $$\left\|f_\ell' - f_n'\right\| \;\leqslant\; \left[\frac{\rho}{V}\right]^n \cdot C \,,$$

for each pair $\ell > n \geqslant K$. Sending $n \nearrow \infty$ sends RhS(17e) $\to 0$.

The limit $\lim_n f_n(0)$ exists, equaling c_0. Now apply the DUC Thm. ♦

A power-series with a new center. We show that a function defined by a PS is analytic in its entire ball-of-convergence.

18: *The setting.* We have a point $P \in \mathbb{C}$ and a sequence $\vec{\mathbf{a}} \subset \mathbb{C}$ such that $\alpha \in (0, +\infty]$, where $\alpha := \mathrm{RoC}(\vec{\mathbf{a}})$. This engenders a $\mathbf{C}^\infty$-fnc from $\mathrm{Bal}_\alpha(P) \to \mathbb{C}$, by

18a: $$\mathcal{F}(z) \;:=\; \sum_{k=0}^{\infty} a_k \cdot [z-P]^k \,.$$

Fix a new center $Q \in \mathbb{C}$ with $|Q-P| < \alpha$. Thus

18b: $\beta \in (0, +\infty]$, *where* $\beta := \alpha - |Q-P|$. *Moreoever,* $\mathrm{Bal}_\beta(Q) \subset \mathrm{Bal}_\alpha(P)$. □

19: New-center theorem. *Take $P, Q, \alpha, \beta, \vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ from (18). For each natnum k, this summation is absolutely convergent:*

19a: $$b_k \;:=\; \sum_{N=k}^{\infty} a_N \cdot \binom{N}{k} \cdot Q^{N-k} \;\in\; \mathbb{C} \,.$$

Moreoever, RoC($\vec{\mathbf{b}}$) $\geqslant \beta > 0$. *This value*

19b: $$\mathcal{G}(z) \;:=\; \sum_{k=0}^{\infty} b_k \cdot [z-Q]^k \,,$$

19c: *agrees with $\mathcal{F}(z)$, for each $z \in \mathrm{Bal}_\beta(Q)$.*

Lastly, for each natnum k,

19d: $$b_k \quad = \quad \tfrac{1}{k!} \cdot \mathcal{F}^{(k)}(Q) \,.$$

In other words, RhS(19b) *is the Taylor series for $\mathcal{F}$, centered at Q.* ◊

Proof. WLOG $P = 0$. Fix a point $Z \in \mathrm{Bal}_\beta(Q)$. Writing $Z = Q + [Z-Q]$, its Nth-power is

$$Z^N \quad = \quad \sum_{k=0}^{N} \binom{N}{k} \cdot Q^{N-k} \cdot [Z-Q]^k \,.$$

Thus, since $Z \in \mathrm{Bal}_\alpha(P)$,

$$\begin{aligned} f(Z) \;&=\; \sum_{N=0}^{\infty} a_N \cdot Z^N \\ &=\; \sum_{N=0}^{\infty} \sum_{k=0}^{N} \underbrace{a_N \cdot \binom{N}{k} \cdot Q^{N-k} \cdot [Z-Q]^k}_{h_{N,k}} \,. \end{aligned}$$

This is a sum, in a certain order, over the set $H := \Big\{ (N,k) \in \mathbb{N}\times\mathbb{N} \;\Big|\; N \geqslant k \Big\}$. We need this sum to be absolutely convergent. The sum $\sum_{N=0}^{\infty}\sum_{k=0}^{N} |h_{N,k}|$ equals

$$*: \quad \sum_{N=0}^{\infty}\sum_{k=0}^{N} |a_N|\cdot\tbinom{N}{k}\cdot|Q|^{N-k}\cdot|Z-Q|^k \;=\; \sum_{N=0}^{\infty} |a_N|\cdot Y^N\,,$$

where $Y := |Q| + |Z-Q|$. From $Z \in \mathrm{Bal}_\alpha(0)$ and (18b), we conclude that $Y < \alpha$. From the proof of Root-test lemma (12, P.5), the righthand side of $(*)$ is finite.

Since $S := \sum_{N=0}^{\infty}\sum_{k=0}^{N} |h_{N,k}|$ is finite, we can reverse the order of summation and conclude that

$$\begin{aligned} S \;&=\; \sum_{k=0}^{\infty}\sum_{N=k}^{\infty} |h_{N,k}| \\ &=\; \sum_{k=0}^{\infty}\Big[\sum_{N=k}^{\infty} |a_N|\cdot\tbinom{N}{k}\cdot|Q|^{N-k}\Big]\cdot|Z-Q|^k\,. \end{aligned}$$

We *could* have chosen our $Z \neq Q$, thus allowing division by $|Z-Q|^k$. Hence, each bracketed sum is finite. So each sum in (19a) is *absolutely convergent*, and we have a well-defined number b_k.

For a general $Z \in \mathrm{Bal}_\alpha(0)$, reversing the original sum gives

$$\begin{aligned} f(Z) \;&=\; \sum_{k=0}^{\infty}\sum_{N=k}^{\infty} h_{N,k} \\ &=\; \sum_{k=0}^{\infty}\Big[\sum_{N=k}^{\infty} a_N\cdot\tbinom{N}{k}\cdot Q^{N-k}\Big]\cdot[Z-Q]^k\,, \end{aligned}$$

which equals $\sum_{k=0}^{\infty} b_k\cdot[Z-Q]^k$.

Establishing (19d)**.** Corollary 17c tells us that

$$k!\cdot b_k \;\overset{\text{by (17c)}}{=\!=}\; \mathcal{G}^{(k)}(Q) \;\overset{\text{by (19c)}}{=\!=}\; \mathcal{F}^{(k)}(Q)\,. \qquad \blacklozenge$$

20**:** Prop'n. *Power-series*

$$*: \qquad \mathcal{F}(z) \;:=\; \sum_{n=0}^{\infty} a_n\cdot[z-Q]^n$$

has positive RoC*. Suppose* $\vec{\mathbf{y}}$ *is a sequence of distinct complex numbers converging to* Q*, such that*

$$\forall j \in \mathbb{Z}_+: \quad \mathcal{F}(y_j) \;=\; 0\,.$$

Then $\vec{\mathbf{a}}$ *is all-zero, and* $\mathcal{F}$ *is the zero function.* $\lozenge$

Proof. WLOG, each $y_j \neq Q$. FTSOC, suppose $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$; let L be the smallest index with $a_L \neq 0$. Formally dividing $(*)$ by $[z-Q]^L$ gives PS

$$\mathcal{G}(z) \;:=\; \sum_{k=0}^{\infty} b_k\cdot[z-Q]^k\,,$$

where each $b_k := a_{L+k}$. Since each $y_j - Q \neq 0$,

$$\mathcal{G}(y_j) \;=\; \mathcal{F}(y_j)/[y_j-Q]^L \;=\; 0\,.$$

But $\mathrm{RoC}(\vec{\mathbf{b}}) = \mathrm{RoC}(\vec{\mathbf{a}}) > 0$, so $\mathcal{G}$ is cts in a nbhd of Q, and thus $\mathcal{G}(Q) = \lim\big(\mathcal{G}(\vec{\mathbf{y}})\big) = 0$. This contradicts that $\mathcal{G}(Q) = b_0 = a_L \neq 0$. $\blacklozenge$

21: PS Uniqueness Thm. *Imagine power-series*

$$\mathcal{F}(z) := \sum_{n=0}^{\infty} a_n \cdot [z-P]^n \quad \text{and}$$
$$\mathcal{G}(z) := \sum_{n=0}^{\infty} b_n \cdot [z-P]^n$$

where $B := \mathsf{BoC}(\vec{\mathbf{a}}) \cap \mathsf{BoC}(\vec{\mathbf{b}})$ *is non-void. Suppose there is a set* $Y \subset B$ *st.* $\mathcal{F}\rfloor_Y = \mathcal{G}\rfloor_Y$, *and* Y *has a cluster point,* Q_0, *in* B. *Then* $\vec{\mathbf{a}} = \vec{\mathbf{b}}$, *so* $\mathcal{F} = \mathcal{G}$. ◊

Remark. It does not suffice for Y to have a cluster-point on the *boundary* of B: Distinct functions $\mathcal{F}(z) := \sin(\frac{1}{z-7})$ and $\mathcal{G} := -\mathcal{F}$ have Taylor series with $\mathsf{RoC} = 7$. Yet

$$\mathcal{F}(y_k) = 0 = \mathcal{G}(y_k)\,, \quad \text{for each posint } k,$$

where $y_k := 7 + \frac{1}{2\pi k}$. □

Proof of (21). Subracting PSes gives us a PS

$$f(z) := \sum_{n=0}^{\infty} c_n \cdot [z-P]^n$$

so that $f\rfloor_Y \equiv 0$, making $\boxed{\vec{\mathbf{c}} \stackrel{?}{=} \vec{\mathbf{0}}}$ our goal.

For each $q \in B := \mathsf{BoC}(\vec{\mathbf{c}})$, let $U(q)$ denote the largest open ball (centered at q) which fits inside B. By the New-center thm, the Taylor-series for f, centered at q, converges to f on all of $U(q)$.

Pick a Y-cluster-point $Q_0 \in B$. By (20), f is identically zero on $U(Q_0)$.

On the line-segment running between Q_0 and P, we can pick a (finite) list of points

$$Q_0, Q_1, \ldots, Q_{K-1}, Q_K := P\,,$$

such that each $Q_k \in U(Q_{k-1})$. Arguing inductively, since f is identically zero on $U(Q_{k-1})$, the the Taylor-series at Q_k has all-zero coeffs. This therefore holds at P. So $\vec{\mathbf{c}} = (0, 0, 0, \ldots)$. ♦

22: Coro. *Suppose* $\mathcal{F}$ *and* $\mathcal{G}$ *are analytic functions on some connected open set* $V \subset \mathbb{C}$. *If*

$$\{z \in V \mid \mathcal{F}(z) = \mathcal{G}(z)\}$$

has a cluster point in V, *then* $\mathcal{F} = \mathcal{G}$. ◊

Abel's theorem

We state this for a PS centered at $Q \in \mathbb{C}$.

23: Abel's Theorem. *Consider a power-series*

$$f(z) := \sum_{n=0}^{\infty} c_n \cdot [z-Q]^n\,,$$

with RoC $\mathcal{R} \in (0, \infty)$, *as well as a point* z_0 *on the circle-of-convergence,* $|z_0 - Q| = \mathcal{R}$. *If* $f(z_0)$ *is finite, then*

$$\lim_{t \nearrow 1} f(t \cdot z_0) = f(z_0)\,.$$

I.e, f *is continuous along radial lines.* ◊

Reduction. WLOG $Q = 0$. WLOG $\mathcal{R} = 1$ and $z_0 = 1$. So

23a: $$f(x) = \sum_{n=0}^{\infty} c_n \cdot x^n$$

with $\mathsf{RoC}(\vec{\mathbf{c}}) = 1$, and $\sum_{n=0}^{\infty} c_n$ finite. Subtract a constant from c_0 so that, WLOG,

23b: $$0 = f(1) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} c_n \stackrel{\text{def}}{=} \lim_{\ell \nearrow \infty} s_\ell\,,$$

where $s_\ell := \sum_{n \in [0\,..\,\ell]} c_n$ is the $\ell^{\textbf{th}}$ ***partial-sum***. Having fixed $\varepsilon > 0$, we will show

$$\forall x \in [\beta, 1): \quad |f(x)| \leqslant 2\varepsilon\,,$$

for a posreal $\beta < 1$ that we will produce. □

Pf. Consider points $x \in J := [0, 1)$. Limit (23b) says we can take $\mathbf{L}$ so large that

$$\forall \ell \geqslant \mathbf{L}: \quad |s_\ell| \leqslant \varepsilon\,.$$

And $\sum_{\ell=\mathbf{L}}^{\infty} s_\ell\, x^\ell$ equals $x^{\mathbf{L}} \cdot \sum_{n=0}^{\infty} s_{n+\mathbf{L}}\, x^n$. So

$$\left| [1-x] \cdot \sum_{\ell=\mathbf{L}}^{\infty} s_\ell\, x^\ell \right| \leqslant x^{\mathbf{L}} \cdot [1-x] \cdot \sum_{n=0}^{\infty} \varepsilon\, x^n$$
$$= x^{\mathbf{L}} \cdot \varepsilon \leqslant \varepsilon\,,$$

since $|x| \leqslant 1$. Let's summarize.

23c: $$\forall x \in J: \quad \left| [1-x] \cdot \sum_{\ell \in [\mathbf{L}\,..\,\infty)} s_\ell\, x^\ell \right| \leqslant \varepsilon\,.$$

Defining β. Wanting $[1-\beta]\cdot\sum_{\ell\in[0\,..\,\mathbf{L})}|s_\ell| < \varepsilon$, simply fix a $\beta<1$ sufficiently close to 1. Thus

$$\text{23d:}\quad \forall x\in[\beta,1):\quad \Bigl|[1-x]\cdot\sum_{\ell\in[0\,..\,\mathbf{L})} s_\ell\, x^\ell\Bigr| \;\leqslant\; \varepsilon\,.$$

Absolute convergence. At a point $x\in J$, series (23a) and $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ are *absolutely convergent*. So we can multiply the series and regroup to conclude:

$$\begin{aligned}\tfrac{1}{1-x}\cdot f(x) &= \Bigl[\sum_{j=0}^{\infty} 1\cdot x^j\Bigr]\cdot\Bigl[\sum_{k=0}^{\infty} c_k x^k\Bigr]\\ &= \sum_{\ell=0}^{\infty}\Bigl[\sum_{j+k=\ell} 1\cdot c_k\Bigr]x^\ell \;\overset{\text{note}}{=\!=}\; \sum_{\ell=0}^{\infty} s_\ell\, x^\ell\,.\end{aligned}$$

For each $x\in[\beta,1)$, then, (23d) and (23c) give

$$|f(x)| \;=\; \Bigl|[1-x]\cdot\sum_{\ell=0}^{\infty} s_\ell\, x^\ell\Bigr| \;\leqslant\; \varepsilon + \varepsilon\,. \quad \blacklozenge$$

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