Multiple ways to empty a tank

Jonathan L.F. King squash@ufl.edu

17 November, 2017 (at *12:02*)

Prolegomenon. A hemispherical tank of radius U :: ft is filled with a fluid of weight-density S :: lb/ft^3. Let $\mathcal{B}$ be the point at the center of the equatorial plane. Let $\mathcal{A}$ be the point on the hemisphere "above" $\mathcal{B}$; so the $\mathcal{AB}$ line-seqment is orthogonal to the equatorial plane.

Evacuation via $\mathcal{A}$, the rounded peak

1a:
$$\begin{aligned} W_{\mathcal{A}} &:= \int_0^U \overbrace{[U-y]}^{\text{Lift}} \cdot S \cdot \boldsymbol{\pi}\overbrace{[U^2-y^2]}^{\text{Radius}^2} \overbrace{\mathrm{d}y}^{\text{Slice height}} \\ &= S{\cdot}\boldsymbol{\pi}\int_0^U [U^3 - U^2y - Uy^2 + y^3]\,\mathrm{d}y \\ &= S{\cdot}\boldsymbol{\pi}\cdot[1-\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}]U^4 = \boldsymbol{\pi}S\cdot\tfrac{5}{12}U^4. \end{aligned}$$

Henceforth, let ***effort***, $\mathbf{E}$, mean $1/\boldsymbol{\pi}$ times work-per-WeightDensity. So $\mathbf{E}_{\mathcal{A}} = W_{\mathcal{A}}/[\boldsymbol{\pi}S]$. Thus,

$$\mathbf{E}_{\text{Full}} > \mathbf{E}_{\mathcal{A}} \overset{\text{recall}}{=\!=} \tfrac{5}{12}U^4, \quad \text{where}$$

$$\mathbf{E}_{\text{Full}} := \overbrace{U}^{\text{Lift}} \cdot \text{Vol}(\substack{\text{Hemi-}\\\text{ball}})/\boldsymbol{\pi} = \tfrac{2}{3}U^4 = \tfrac{8}{12}U^4.$$

Evacuation via $\mathcal{B}$, the equatorial plane

Let's set up this integral in several ways. Firstly,

2: $\mathbf{E}_{\mathcal{B}} = \mathbf{E}_{\text{Full}} - \mathbf{E}_{\mathcal{A}} = [\tfrac{8}{12} - \tfrac{5}{12}]U^4 = \tfrac{1}{4}U^4.$

Or, –with the origin at the sphere's center– integrating "with disks" gives

2a:
$$\begin{aligned} \mathbf{E}_{\mathcal{B}} &= \int_0^U \overbrace{y}^{\text{Lift}} \cdot \overbrace{[U^2-y^2]}^{\text{Radius}^2} \overbrace{\mathrm{d}y}^{\text{Slice height}} \\ &= [\tfrac{1}{2}-\tfrac{1}{4}]U^4 \overset{\text{note}}{=\!=} \tfrac{1}{4}U^4. \end{aligned}$$

Alternatively, putting the origin at $\mathcal{A}$,

2b:
$$\begin{aligned} \mathbf{E}_{\mathcal{B}} &= \int_0^U \overbrace{U-z}^{\text{Lift}} \cdot \overbrace{[U^2-[U-z]^2]}^{\text{Radius}^2} \overbrace{\mathrm{d}z}^{\text{Slice height}} \\ &= \int_0^U [2U^2y - 3Uy^2 + y^3]\,\mathrm{d}z \\ &= [2{\cdot}\tfrac{1}{2} - 3{\cdot}\tfrac{1}{3} + \tfrac{1}{4}]U^4 \overset{\text{note}}{=\!=} \tfrac{1}{4}U^4. \end{aligned}$$

Using polar coordinates, note that $y = U\sin(\theta)$. Thus $\frac{\mathrm{d}y}{\mathrm{d}\theta} = U\cos(\theta)$, so $\mathrm{d}y = U\cos(\theta)\,\mathrm{d}\theta$. Hence

2c:
$$\begin{aligned} \mathbf{E}_{\mathcal{B}} &= \int_0^{\frac{\pi}{2}} \overbrace{U\sin(\theta)}^{\text{Lift}} \cdot \overbrace{U^2{\cdot}\cos(\theta)^2}^{\text{Radius}^2} \cdot \overbrace{U\cos(\theta)\,\mathrm{d}\theta}^{\text{Slice height}} \\ &= U^4\cdot\left[\tfrac{-1}{4}\cos(\theta)^4\right]\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= U^4\cdot\tfrac{1}{4}\cdot[\cos(0)^4 - \cos(\tfrac{\pi}{2})^4] \overset{\text{note}}{=\!=} \tfrac{1}{4}U^4. \end{aligned}$$

Trickier, is integrating w.r.t the radius –let's call it "x"– of the cross-sectional disks. Note that y equals $[U^2-x^2]^{1/2}$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}[U^2-x^2]^{-1/2}\cdot[\text{-}2x]$. Thus

$$\text{-}\mathrm{d}y = \frac{x}{\sqrt{U^2-x^2}}\,\mathrm{d}x\,.$$

Why did I write *negative* $\mathrm{d}y$? Well, as x increases, note that y decreases. Since I'm integrating w.r.t x, the quantity $\text{-}\mathrm{d}y$ measures that width *positively.* Hence

2d:
$$\begin{aligned} \mathbf{E}_{\mathcal{B}} &= \int_{x=0}^{x=U} \overbrace{\sqrt{U^2-x^2}}^{\text{Lift}} \cdot \overbrace{x^2}^{\text{Radius}^2} \cdot \overbrace{[\text{-}\mathrm{d}y]}^{\text{Slice height}} \\ &= \int_0^U \sqrt{U^2-x^2}\cdot x^2 \cdot \overbrace{\tfrac{x}{\sqrt{U^2-x^2}}\,\mathrm{d}x}^{\text{Slice height}} \\ &= \int_0^U x^3\,\mathrm{d}x \overset{\text{note}}{=\!=} \tfrac{1}{4}U^4. \end{aligned}$$

Using cylindrical shells. All the preceding can be viewed as simple change-of-variables (substitution) of a single integral, since they all slice the same way. But now, let's slice with a cylindrical-saw.

The cylinder at x has height $\sqrt{U^2-x^2}$, and so its CoM ("Center of Mass") is at half that height, by symmetry. The circumference of the shell is $2\boldsymbol{\pi}x$, so

$$\boldsymbol{\pi}\mathbf{E}_{\mathcal{B}} = \int_0^U \overbrace{\tfrac{1}{2}\sqrt{U^2-x^2}}^{\text{Lift}} \cdot \overbrace{[2\boldsymbol{\pi}x]{\cdot}\sqrt{U^2-x^2}}^{\text{Area}} \overbrace{\mathrm{d}x}^{\text{Thickness}}. \quad \textit{So}$$

2e:
$$\mathbf{E}_{\mathcal{B}} = \int_0^U x\cdot[U^2-x^2]\,\mathrm{d}x \overset{\text{note}}{=\!=} \tfrac{1}{4}U^4,$$

as we saw in (2a).

Filename: `Problems/Analysis/Calculus/tank-evacuation.latex`

As of: *Thursday 27Aug2015.* Typeset: *17Nov2017 at 12:02.*