Differential Eqns MAP2302 3228 | **T-Class** | Prof. JLF King Wedn., 17Oct2018

T1: *Show no work.*

a Prof. King thinks that submitting a ROBERT LONG PRIZE ESSAY [typically 2 prizes, $500 total] is a *really good idea*, and the due date for the emailed-PDF is typically mid-March. Circle:

Yes **True** ***Résumé material!***

b Bacteria with birth-multiplier $\mathbf{B} :: \frac{1}{\mathsf{min}}$ are in a petri dish with carrying capacity $\mathbf{C} :: \mathsf{oz}$. The population, $p(t) :: \mathsf{oz}$, satisfies the Logistic DE.

The DE is $p'(t) = \mathbf{B}\cdot\left[1 - \frac{p(t)}{\mathbf{C}}\right]\cdot p(t)$.

For *Hysteria* bacteria, $\mathbf{B} = \frac{1/5}{\mathsf{min}}$. This petri dish has $\mathbf{C}$=16oz, with initial population $\mathbf{p}_0 = 2\mathsf{oz}$. The time when *Hysteria* has reached half the carrying capacity

is ______ min $\approx$ ______ min.

[NB: You may use exp() and log() to express your answer.]

Hysterical Soln: DfyNotes showed that

$$p(t) = \frac{\mathbf{C}}{1 + [\frac{\mathbf{C}}{\mathbf{p}_0} - 1]\cdot \mathsf{e}^{-\mathbf{B}\cdot t}}.$$

So we seek the time, τ, when $[\frac{\mathbf{C}}{\mathbf{p}_0} - 1]\cdot\mathsf{e}^{-\mathbf{B}\cdot\tau} = 1$, i.e $\mathsf{e}^{\mathbf{B}\cdot\tau} = \frac{\mathbf{C}}{\mathbf{p}_0} - 1$. Thus

$$\tau = \frac{1}{\mathbf{B}}\cdot\log\left(\frac{\mathbf{C}}{\mathbf{p}_0} - 1\right) = 5\,\mathsf{min}\cdot\log\left(\frac{16\,\mathsf{oz}}{2\,\mathsf{oz}} - 1\right) = 5\cdot\log(7)\cdot\mathsf{min}.$$

Note $2.7 < \mathsf{e}$, so $7.29 = [2.7]^2 < \mathsf{e}^2$. Thus $7 \overset{\text{little bit}}{<} \mathsf{e}^2$. I.e, log(7) is bit less than 2. So $\tau \approx 5\cdot 2\,\mathsf{min} = 10\,\mathsf{min}$.

ESTIMATING τ *without* SOLVING THE DE: In 5min, the Malthusian model multiplies the population by e (i.e continuous compounding). Thus $\mathsf{e}\cdot 2\,\mathsf{oz} < 2.8\cdot 2\,\mathsf{oz} = 5.6\,\mathsf{oz} \overset{\text{note}}{<} 8\,\mathsf{oz} = \frac{\mathbf{C}}{2}$. So $\boxed{\tau > 5\mathsf{min}}$.

For Logistic model @**time-zero**, the birth-rate, $p'(0\mathsf{min})$, is

$$\mathbf{B}\cdot\left[1 - \frac{\mathbf{p}_0}{\mathbf{C}}\right]\cdot\mathbf{p}_0 = \mathbf{B}\cdot\left[1 - \frac{2\,\mathsf{oz}}{16\,\mathsf{oz}}\right]\cdot 2\,\mathsf{oz} = \frac{7}{20}\,\frac{\mathsf{oz}}{\mathsf{min}}.$$

If the new bacteria were sterile, then B-R stays at $\frac{7}{20}\frac{\mathsf{oz}}{\mathsf{min}}$. We need to increase the pop. by $\frac{\mathbf{C}}{2} - \mathbf{p}_0 = 6\mathsf{oz}$. Hence the time, $\mathbf{t}_{\text{sterile}}$, needed in this sterile model is

$$\frac{7}{20}\,\frac{\mathsf{oz}}{\mathsf{min}}\cdot\mathbf{t}_{\text{sterile}} = 6\mathsf{oz}, \quad \textit{i.e}$$
$$\mathbf{t}_{\text{sterile}} = \frac{120}{7}\mathsf{min} \overset{\text{note}}{<} 17.2\,\mathsf{min}.$$

Thus, $\boxed{5\,\mathsf{min} < \tau < 17.2\,\mathsf{min}}$.

The actual value is

$$\tau = 5\cdot\log(7)\cdot\mathsf{min} \approx 9.73\,\mathsf{min}.$$

ASIDE: Had we been willing, in the Malthusian model [MM], to estinate logarithm, then we would have gotten a better lower-bound for τ. DfyNotes showed that the doubling-time for MM is $\frac{\log(2)}{\mathbf{B}} = \log(2)\cdot 5\,\mathsf{min}$. Doubling twice will carry $\mathbf{p}_0$ to $8\,\mathsf{oz}$, so the MM-time is

$$\log(2)\cdot 10\,\mathsf{min} \overset{\text{note}}{>} 6.9\,\mathsf{min}.$$

Hence $\boxed{6.9\,\mathsf{min} < \tau < 17.2\,\mathsf{min}}$.

c Fnc $y_{\alpha,\beta}(t) = \alpha e^{At} + \beta e^{Bt} + P\cdot\sin(t) + Q\cdot\cos(t)$ is the general soln to

$$*: \qquad 3y'' + 4y' + y = \cos(t), \quad \textit{with numbers}$$

$A=$ ______, $B=$ ______, $P=$ ___, $Q=$ ___.

Also, the *constants* on LhS($*$) are $3, 4, 1$. With the DE describing the position of a spring, the *constant* corresponding to Hooke's constant is 1.

Soln: Operator $\mathsf{L} := 3\mathbf{D}^2 + 4\mathbf{D} + \mathbf{I}$ has aux-poly $f(z) := 3z^2 + 4z + 1$. So $\text{Discr}(f) = 4^2 - 4\cdot3\cdot1 = 4$. Thus the roots of f are

$$A := \tfrac{1}{6}[-4 + \sqrt{4}] = \frac{-1}{3} \quad \textit{and} \quad B := \tfrac{1}{6}[-4 - \sqrt{4}] = -1;$$

they satisfy that $\mathsf{L}(e^{At}) = 0 = \mathsf{L}(e^{Bt})$.

With $\boldsymbol{\mathcal{C}} := \cos(t)$ and $\boldsymbol{\mathcal{S}} := \sin(t)$, we seek numbers P, Q st. $\mathsf{L}(P\boldsymbol{\mathcal{S}} + Q\boldsymbol{\mathcal{C}}) = 0\cdot\boldsymbol{\mathcal{S}} + 1\cdot\boldsymbol{\mathcal{C}}$. Computing,

$$\begin{aligned} \mathbf{I}(P\boldsymbol{\mathcal{S}} + Q\boldsymbol{\mathcal{C}}) &= P\boldsymbol{\mathcal{S}} + Q\boldsymbol{\mathcal{C}} \quad \textit{and} \\ \mathbf{D}(P\boldsymbol{\mathcal{S}} + Q\boldsymbol{\mathcal{C}}) &= -Q\boldsymbol{\mathcal{S}} + P\boldsymbol{\mathcal{C}} \quad \textit{and} \\ \mathbf{D}^2(P\boldsymbol{\mathcal{S}} + Q\boldsymbol{\mathcal{C}}) &= -P\boldsymbol{\mathcal{S}} - Q\boldsymbol{\mathcal{C}}. \end{aligned}$$

Hence $\mathsf{L}(P\boldsymbol{\mathcal{S}} + Q\boldsymbol{\mathcal{C}})$ equals

$$[-2P + -4Q]\boldsymbol{\mathcal{S}} + [4P + -2Q]\boldsymbol{\mathcal{C}}.$$

Eqn-pair $\boxed{-2P + -4Q = 0}$ and $\boxed{4P + -2Q = 1}$ yields $P = 1/5$ and $Q = -1/10$.

d Operators $\mathsf{V}, \mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ map from $\mathbf{C}^\infty \to \mathbf{C}^\infty$, and V is linear. The other maps are

$$\mathbf{P}(f) := [t \mapsto f(t) + 3], \quad \mathbf{Q}(f) := [t \mapsto f(t+3)],$$
$$\mathbf{R}(f) := [t \mapsto f(f(t))], \quad \mathbf{S}(f) := \mathsf{V}(\mathsf{V}(f)),$$

Then... **P** *is linear:* *T* *F*. **Q** *is linear:* *T* *F*.
R *is linear:* *T* *F*. **S** *is linear:* *T* *F*.

Linear thinking: Map L is ***linear*** if $\forall f,g$ and $\forall_{\text{scalars}}\,\alpha$: $\mathsf{L}(f+g) = \mathsf{L}(f) + \mathsf{L}(g)$ and $\mathsf{L}(\alpha f) = \alpha \mathsf{L}(f)$.
Combining: $\forall f,g$ *and* $\forall \alpha,\beta$: $\mathsf{L}(\alpha f + \beta g) = \alpha \mathsf{L}(f) + \beta \mathsf{L}(g)$.

Operators **Q** and **S** satisfy these linearity conditions. E.g, $\mathbf{S}(\alpha f + \beta g)$ equals

$$\begin{aligned}\mathsf{V}(\mathsf{V}(\alpha f + \beta g)) &= \mathsf{V}(\alpha\,\mathsf{V}(f) + \beta\,\mathsf{V}(g)), \text{ since } \mathsf{V} \text{ is linear,}\\ &= \alpha\,\mathsf{V}(\mathsf{V}(f)) + \beta\,\mathsf{V}(\mathsf{V}(f)), \text{ since } \mathsf{V} \text{ is linear.}\end{aligned}$$

This last indeed equals $\alpha\mathbf{S}(f) + \beta\mathbf{S}(g)$.

How about **P**? Well $\mathbf{P}(\alpha f + \beta g) \stackrel{\text{def}}{=} [\alpha f + \beta g] + 3$. OTOHand, $\alpha\mathbf{P}(f) + \beta\mathbf{P}(g) \stackrel{\text{def}}{=} \alpha[f+3] + \beta[g+3]$, which equals $[\alpha f + \beta g] + 3[\alpha + \beta]$. So $\mathbf{P}(\alpha f + \beta g) \stackrel{?}{=} \alpha\mathbf{P}(f) + \beta\mathbf{P}(g)$ IFF $3 \stackrel{?}{=} 3[\alpha + \beta]$, i.e, IFF $\alpha + \beta = 1$. THE UPSHOT: Our **P** is an *affine* operator, but not a *linear* operator.

Whereas **P** was almost linear, our **R** is wildly non-linear. E.g, let h be the identity fnc $h(t) := t$. Now

$$[\mathbf{R}(5h)](t) \stackrel{\text{def}}{=} [5h]([5h](t)) = 5 \cdot h(5 \cdot h(t)) = 5^2 t.$$

OTOHand, $[5\,\mathbf{R}(h)](t) = 5 \cdot [\mathbf{R}(h)](t) = 5 \cdot h(h(t)) = 5t$. But does $5^2 t = 5t$ hold for *all* t? *No!* I.e $[t \mapsto 5^2 t]$ and $[t \mapsto 5t]$ are *different* fncs.

e With $v := \exp(-2 + 5\boldsymbol{i})$, then $|v| =$ ....

This $|v|$ lies in circle the correct interval

$[0, \frac{1}{2})$, $[\frac{1}{2}, 1)$, $[1, 2)$, $[2, 4)$, $[4, 8)$, $[8, \infty)$.

$\mathbb{C}$-arithmetic: Recall $\mathsf{e}^{-2+5\boldsymbol{i}} = \mathsf{e}^{-2} \cdot \mathsf{e}^{5\boldsymbol{i}}$ and $|\mathsf{e}^{5\boldsymbol{i}}| = 1$. Hence $|v| = \mathsf{e}^{-2}$. As $3 > \mathsf{e} > 2$, then $\frac{1}{9} < |v| < \frac{1}{4}$.

T2: *Show no work.*

i A tank initially has 80gal of salinity $2\frac{\text{lb}}{\text{gal}}$ brine. Pipe-1 feeds the tank, at rate $3\frac{\text{gal}}{\text{min}}$, with salinity $1\frac{\text{lb}}{\text{gal}}$ brine. Pipe-2 feeds at $2\frac{\text{gal}}{\text{min}}$ with salinity $2\frac{\text{lb}}{\text{gal}}$. The tank discharges brine at $9\frac{\text{gal}}{\text{min}}$. Until the tank empties, it holds

$W(t)=$ [______] gal; it empties in ______ min.

The amount, $y(t)$, of lb of salt in the tank at time t, satisfies FOLDE $\frac{dy}{dt} + C(t)\cdot y = G(t)$, where

$C(t)=$ ______ and $G(t)=$ ______.

Brine Soln: The net brine outflow, in gal/min, is $9-[3+2]=4$. Hence $W(t) = \left[80 - \frac{4t}{\text{min}}\right]\text{gal}$. Thus the tank empties in $\frac{80}{4}\text{min} = 20\text{min}$.

From the pipe-1 and pipe-2 data,

$$y' = \overbrace{[\mathrm{R}_1\cdot\sigma_1] + [\mathrm{R}_2\cdot\sigma_2]}^{\text{Input}} - \overbrace{\rho\cdot\frac{y}{W(t)}}^{\text{Output}}.$$

In FOLDE form,

$$y' + \frac{\rho}{W(t)}y = [\mathrm{R}_1\sigma_1] + [\mathrm{R}_2\sigma_2].$$

Hence

$$C(t) = \frac{\rho}{W(t)} = \frac{9}{80\text{min} - 4t}, \quad \textit{and}$$

$$G(t) = [\mathrm{R}_1\sigma_1] + [\mathrm{R}_2\sigma_2] = [3\cdot 1 + 2\cdot 2]\frac{\text{lb}}{\text{min}} = 7\frac{\text{lb}}{\text{min}}.$$

ii A *critically-damped* unforced spring has DE

$$*: \quad \mathbf{M}y'' + \mathbf{B}y' + \mathbf{K}y = 0\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}, \quad \textit{where}$$
$$\mathbf{M} := 3\text{kg}, \textit{ and the Hooke's constant is } \mathbf{K} := 75\frac{\text{kg}}{\text{sec}^2}.$$

The damping constant $\mathbf{B}= 2\sqrt{\mathbf{MK}} = 2\cdot 3\cdot 5\frac{\text{kg}}{\text{sec}} = 30\frac{\text{kg}}{\text{sec}}$.

The *general soln* to critically-damped $(*)$ is

$y(t)=\left[\alpha\cdot \text{______} + \beta\cdot \text{______}\right]\text{m}$.

Here, $\alpha,\beta \in \mathbb{C}$. (*The 3 blanks will have units & numbers in various places.* Maybe exp(?) is more convenient than $\mathrm{e}^?$ notation.) The **specific** soln with $y(0\text{sec}) = 0\text{m}$ and $y'(0\text{sec}) = 2\frac{\text{m}}{\text{sec}}$ has

$\alpha=$ ______, $\beta=$ ______.

Damp Soln: Aux.poly $\mathbf{M}z^2 + \mathbf{B}z + \mathbf{K}$ has a double-root of $\mathbf{r} := \frac{-\mathbf{B}}{2\mathbf{M}} = -5/\text{sec}$. Thus the general soln to $(*)$ is

$$y(t) = \left[\alpha\cdot \mathrm{e}^{\mathbf{r}t} + \beta\cdot\frac{t}{\text{sec}}\mathrm{e}^{\mathbf{r}t}\right]\text{m}$$
$$= \left[\alpha\cdot\exp\left(\frac{-5}{\text{sec}}t\right) + \beta\cdot\frac{t}{\text{sec}}\exp\left(\frac{-5}{\text{sec}}t\right)\right]\text{m},$$

where α,β are dimensionless.

Note $0\text{m} = y(0\text{sec}) = \alpha\text{m}$, so $\alpha = 0$. Hence $y'(t)$ equals $\beta\cdot[1+t\mathbf{r}]\,\mathrm{e}^{\mathbf{r}t}\cdot\frac{\text{m}}{\text{sec}}$. Thus

$$2\frac{\text{m}}{\text{sec}} = y'(0\text{sec}) = \beta\frac{\text{m}}{\text{sec}}. \quad \textit{So } \beta = 2.$$

T1:	___ ___ ___	115pts
T2:	___ ___	70pts
Total:	___ ___ ___	185pts