Linear Algebra: Some things to know

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Entrance. Below is a subset of the definitions, algorithms and theorems from class.

For each defn, you should know several examples where the objects fulfill the defn, and several examples where the objects *fail* the defn. [E.g, what is an example of a set of vectors that is *not* L.I. What is an example of a pair of VSes and a map T between them that: *is* linear? *fails* to be linear? –what specific axiom fails?]

Definition. A ***commutative group*** is a triple $(\mathbf{V}, +, \mathbf{0})$ where $\mathbf{V}$ is a set, $+$ is a binary operation $\mathbf{V} \times \mathbf{V} \to \mathbf{V}$, and $\mathbf{0} \in \mathbf{V}$, satisfying the following axioms for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbf{V}$:

CG1: Element $\mathbf{0}$ is an ***identity element*** for "+", i.e,

$$\mathbf{x} + \mathbf{0} = \mathbf{x} = \mathbf{0} + \mathbf{x}.$$

CG2: Addition is associative: $\mathbf{x}+[\mathbf{y}+\mathbf{z}] = [\mathbf{x}+\mathbf{y}]+\mathbf{z}$.

CG3: Every element $\mathbf{x}$ has an ***additive inverse*** $\mathbf{x}'$ satisfying: $\mathbf{x} + \mathbf{x}' = \mathbf{0} = \mathbf{x}' + \mathbf{x}$.

CG4: Addition is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

[Note: A ***group*** $(\mathbf{V}, +, \mathbf{0})$, but which is not necessarily commutative, is required to satisfy (CG 1–3) but is not required to satisfy (CG 4).] □

It is an easy theorem that $\mathbf{0}$ is the unique identity element for "+" and that additive inverses are unique. The additive inverse of $\mathbf{x}$ is usually written as "$-\mathbf{x}$".

Definition. A ***vectorspace*** is a five-tuple

$$(\mathbf{V}, +, \mathbf{0}, \mathbf{F}, \cdot),$$

where $\mathbf{V}$ is a *set* [of vectors], with $\mathbf{0} \in \mathbf{V}$, where $\mathbf{F}$ is a field, and where the ***scalar-vector multiplication*** operation $\cdot$ is a map $\mathbf{F} \times \mathbf{V} \to \mathbf{V}$.

Using ***SVM*** to abbreviate ***scalar-vector multiplication*** [usually called "scalar multiplication"], the five-tuple must satisfy the following.

For all $\alpha, \beta \in \mathbf{F}$ *and* $\mathbf{x},\mathbf{y} \in \mathbf{V}$:

SV1: Triple $(\mathbf{V}, +, \mathbf{0})$ is a commutative group.

SV2: The SVM distributes over *vector* addition: $\alpha[\mathbf{x} + \mathbf{y}] = \alpha\mathbf{x} + \alpha\mathbf{y}$.

Also, SVM distributes over *scalar* addition: $[\alpha + \beta]\mathbf{x} = \alpha\mathbf{x} + \alpha\mathbf{y}$.

SV3: Multiplication associates with SVM, i.e $[\alpha \cdot \beta]\mathbf{x} = \alpha \cdot [\beta\mathbf{x}]$.

SV4: Scalars 1 and 0 act as follows: $1 \cdot \mathbf{x} = \mathbf{x}$ and $0 \cdot \mathbf{x} = \mathbf{0}$. □

Know the following terms: The ***cardinality*** of a set S is the number of elements in S, and written $|S|$, or sometimes $^{\#}S$. A ***subspace*** of a vectorspace. [Recall that the ***trivial subspace*** $\{\mathbf{0}\}$ is the unique 0-dimensional subspace. Recall that the emptyset, $\varnothing = \{\}$, is *not* a vectorspace, because it has no identity element.]

A ***linear combination*** of a set of vectors. The ***span*** of a set of vectors. [Recall that the span of a set of vectors in $\mathbf{V}$ is always a subspace of $\mathbf{V}$. Recall that $\mathrm{Spn}(\varnothing) = \{\mathbf{0}\}$.] Recall that a collection $S \subset \mathbf{V}$ is ***linearly independent*** if the only linear combination of vectors in S which equals $\mathbf{0}$, is the ***trivial combination***, that is, the combination where all scalars are 0. A ***basis*** for $\mathbf{V}$ is a linearly independent subset of $\mathbf{V}$ which spans $\mathbf{V}$.

Some important theorems. In a vectorspace $\mathbf{V}$:

1: Theorem. *Every vectorspace has a basis. Each linearly-independent set can be extended to (i.e, is a subset of) a basis. Each spanning set can be cut down to (i.e, is a superset of) a basis.* ◊

2: Theorem. *The cardinality of every spanning set is greater-equal the cardinality of every linearly-independent set. In particular, each two bases have the same cardinality; this number is called the* ***dimension*** *of* $\mathbf{V}$. ◊

Terms and algorithms. Know the definitions of the following terms, and how to perform the following algorithms:

"Algorithm". "Augmented matrix". Know the three "elementary row operations", and what "row equivalence" is. Be able to precisely describe the Gaussian Elimination algorithm. "Reduced row-echelon form". A "pivot" position. "Free column". A "consistent" system of linear equations. Know how to compute the "solution set" to a system of linear equations or to a vector equation $A\mathbf{x} = \mathbf{b}$ [where A is a $k{\times}n$ matrix, $\mathbf{b} \in \mathbf{F}^k$ is known, and $\mathbf{x} \in \mathbf{F}^n$ is the unknown], and how to describe the solution set parametrically. Recall that such a solution set is either empty, or is a translated vector subspace of $\mathbf{F}^n$, ie "an ***affine subspace***" or "a ***flat***".

The "column-span" and "row-span" of a matrix, as well as the "column-rank" and "row-rank", and know how to compute these four things.

"Linear transformation". The "inverse" of a linear transformation. The "inverse of an invertible square matrix", and how to compute it. Know how to compute the matrix corresponding to a given linear transformation.

"Change-of-basis matrix" and how to compute such.

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