An obstruction to K-fold splitting

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ABSTRACT. For a transformation T, if the sum of the K-th root of its partial mixing with the K-th root of its partial rigidity exceeds 1, then the transformation can have no factor isomorphic to a K-fold cartesian product.

The inspiration for this note is Nat Friedman's result, [1], that a transformation cannot be a cartesian product if its partial rigidity and partial mixing sum exactly to one, even along a subsequence.

Say that transformation $T: X \to X$ ***K-fold splits*** if T is a K-fold cartesian product $S_1 \times \cdots \times S_K$ where none of the S_k live on a 1-point space. [Our context is that of bi-measure preserving maps of a Lebesgue probability space.] We now define the notions of partial rigidity and mixing. Given a sequence of integers $\vec{s} = \{s[k]\}_{k=1}^{\infty}$ going to infinity, define four quantities

$$\mathbf{m}(T;\vec{s}) := \inf_{A,B} \frac{1}{\mu(A)\mu(B)} \liminf_{k\to\infty} \mu\big(A \cap T^{-s[k]}B\big) \qquad \mathbf{r}(T;\vec{s}) := \inf_{A} \frac{1}{\mu(A)} \liminf_{k\to\infty} \mu\big(A \cap T^{-s[k]}A\big)$$

$$\mathbf{M}(T;\vec{s}) := \inf_{A,B} \frac{1}{\mu(A)\mu(B)} \limsup_{k\to\infty} \mu\big(A \cap T^{-s[k]}B\big) \qquad \mathbf{R}(T;\vec{s}) := \inf_{A} \frac{1}{\mu(A)} \limsup_{k\to\infty} \mu\big(A \cap T^{-s[k]}A\big)$$

where the above infimums are taken over all sets $A, B \subset X$ of positive measure. When T is understood, we suppress T and write $\mathbf{m}(\vec{s})$ for $\mathbf{m}(T;\vec{s})$. Say that sequence $\vec{n}$ is an (eventual) ***subsequence*** of $\vec{s}$, written $\vec{n} \prec \vec{s}$, if after discarding finitely many terms from $\vec{n}$ the resulting sequence is an actual subset of $\vec{s}$.

The quantity $\mathbf{m}(T;\vec{s})$ is called the ***partial mixing*** of T along $\vec{s}$ and is also written as $\operatorname{mix}(T;\vec{s})$. For T, the ***partial rigidity*** along $\vec{s}$ is

$$\operatorname{rig}(T;\vec{s}) := \sup_{\vec{n}:\vec{n}\prec\vec{s}} \mathbf{r}(T;\vec{n}).$$

In both the above, when $\vec{s} = \mathbb{N}$ we write $\operatorname{mix}(T)$ and $\operatorname{rig}(T)$, respectively.

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Note. If $\mathcal{D}$ is any subcollection which is dense (symmetric-difference metric) in the collection of measurable sets, then none of the four quantities above would change were the infimums computed over all $A, B \in \mathcal{D}$ rather than over all measurable A and B.

We will use "stable subsequence $\vec{n}$" in the following shorthand: *"There exists a stable subsequence $\vec{n} \prec \vec{s}$ such that* Property$(\vec{n}, \vec{s})$" shall mean for any further subsequence $\vec{m} \prec \vec{n}$ that Property$(\vec{m}, \vec{s})$ holds.

PROPOSITION. *Given any transformation T and sequence $\vec{s}$.*

(a) $0 \leqslant \mathbf{m}(\vec{s}) \leq \mathbf{M}(\vec{s}) \leq 1$ *and* $0 \leqslant \mathbf{r}(\vec{s}) \leq \mathbf{R}(\vec{s}) \leq 1$.

(b) *If $\vec{n} \prec \vec{s}$ then:*

$$\mathbf{m}(\vec{n}) \geq \mathbf{m}(\vec{s}); \qquad \mathbf{r}(\vec{n}) \geq \mathbf{r}(\vec{s});$$
$$\mathbf{M}(\vec{n}) \leq \mathbf{M}(\vec{s}); \qquad \mathbf{R}(\vec{n}) \leq \mathbf{R}(\vec{s}).$$

(c) *If X is not a 1-point space:* $1 \geq \mathbf{M}(\vec{s}) + \mathbf{r}(\vec{s}), \qquad 1 \geq \mathbf{m}(\vec{s}) + \mathbf{R}(\vec{s})$.

(d) *There exists a stable subsequence $\vec{n} \prec \vec{s}$, such that $\mathbf{M}(\vec{n}) = \mathbf{m}(\vec{n})$ and $\mathbf{R}(\vec{n}) = \mathbf{r}(\vec{n})$.*

(e) *There exists a stable subsequence $\vec{n} \prec \vec{s}$ for which $\mathbf{r}(T; \vec{n}) = \mathrm{rig}(T; \vec{s})$.*

(f) *Suppose T is a cartesian product $S_1 \times \ldots \times S_K$. Then $\mathbf{r}(T; \vec{s}) \geq \mathbf{r}(S_1; \vec{s}) \cdot \ldots \cdot \mathbf{r}(S_K; \vec{s})$ and $\mathbf{R}(T; \vec{s}) \leq \mathbf{R}(S_1; \vec{s}) \cdot \ldots \cdot \mathbf{R}(S_K; \vec{s})$. Moreover, there exists a stable subsequence $\vec{n} \prec \vec{s}$ for which*

$$\mathbf{r}(S_1 \times \ldots \times S_K; \vec{n}) = \mathbf{r}(S_1; \vec{n}) \cdot \ldots \cdot \mathbf{r}(S_K; \vec{n})$$

with the parallel assertion for $\mathbf{R}$. The analogous (in)equalities hold when $\mathbf{r}$ and $\mathbf{R}$ are replaced by $\mathbf{m}$ and $\mathbf{M}$.

Proof of (c). The argument for the second inequality is similar to that of the first and so we argue the first: In light of $\mu(A \cap T^{-k} A^c) = \mu(A) - \mu(A \cap T^{-k} A)$, we have that for any non-trivial A

$$\begin{aligned}
\mathbf{M}(\vec{s}) &\leq \frac{1}{\mu(A)\mu(A^c)} \limsup_{k\to\infty} \mu(A \cap T^{-s[k]} A^c) \\
&= \frac{1}{\mu(A)\mu(A^c)} \left[\mu(A) - \liminf_{k\to\infty} \mu(A \cap T^{-s[k]} A)\right] \\
&\leq \frac{1}{\mu(A^c)} \left[1 - \mathbf{r}(\vec{s})\right].
\end{aligned}$$

If the space has sets of arbitrarily small positive measure, then send $\mu(A) \to 0$ and conclude that $\mathbf{M}(\vec{s}) \leq 1 - \mathbf{r}(\vec{s})$. Or, if $\mathbf{r}(\vec{s})$ equals zero, we are still done since always $\mathbf{M}(\vec{s}) \leq 1$.

On the other hand, if we cannot send $\mu(A)$ to zero then the space is purely atomic and, since $\mathbf{r}(\vec{s}) > 0$, there is a non-trivial atom $x \in X$ such that $T^{-s[k]} x = x$ for all large k. Setting $A := \{x\}$ and $B := X \setminus \{x\}$ shows $\mathbf{M}(\vec{s})$ to be zero. ♦

Proof of (d). We prove that $\mathbf{M}(\vec{n}) = \mathbf{m}(\vec{n})$. Given an ε, pick sets A, B so that

$$\liminf_{k\to\infty} \mu(A \cap T^{-s[k]} B) < \left[\mathbf{m}(\vec{s}) + \varepsilon\right] \mu(A)\mu(B).$$

Let $\vec{v}$ be a subsequence of $\vec{s}$ such that $\lim_k \mu(A \cap T^{-v[k]} B)$ exists and equals the above liminf. Thus

$$\mathbf{M}(\vec{v}) \leq \mathbf{m}(\vec{s}) + \varepsilon \leq \mathbf{m}(\vec{v}) + \varepsilon. \tag{1}$$

Now pick some $\varepsilon_j \searrow 0$. Use (1) to inductively pick subsequences $\vec{s} \supset \vec{v_1} \supset \vec{v_2} \supset \cdots$ such that $\mathbf{M}(\vec{v_j}) \leq \mathbf{m}(\vec{v_j}) + \varepsilon_j$. Define $\vec{n}$ by $n[k] \coloneqq v_k[k]$. Since $\vec{n}$ is an eventual subsequence of each $\vec{v_j}$

$$\mathbf{m}(\vec{n}) \leq \mathbf{M}(\vec{n}) \leq \mathbf{M}(\vec{v_j}) \leq \mathbf{m}(\vec{v_j}) + \varepsilon_j \leq \mathbf{m}(\vec{n}) + \varepsilon_j.$$

Sending $j \to \infty$ achieves the first equality of (d). Evidently this equality is stable since $\mathbf{M}$ and $\mathbf{m}$ move in opposite directions under subsequencing.

A similar argument shows the existence of a subsequence $\vec{m} \prec \vec{s}$ for which the second equality, $\mathbf{R}(\vec{m}) = \mathbf{r}(\vec{m})$, holds. Picking an $\vec{n} \prec \vec{m}$ so that $\mathbf{M}(\vec{n}) = \mathbf{m}(\vec{n})$ gives us both equalities simultaneously. ♦

Proof of (e). Let $\mathcal{D} = \{A_j\}_{j=1}^{\infty}$ be a dense collection of sets. Pick $\varepsilon_j \searrow 0$ and subsequence $\vec{v_j} \subset \vec{s}$ such that

$$\mathbf{r}(\vec{v_j}) > \mathrm{rig}(\vec{s}) - \varepsilon_j.$$

Fix J. Let $m \coloneqq v_J[k]$ for a k sufficiently large that

$$\forall j < J: \quad \frac{1}{\mu(A_j)} \mu\big(A_j \cap T^{-m} A_j\big) > \mathrm{rig}(\vec{s}) - \varepsilon_J.$$

Define $\vec{n}$ inductively by setting $n[J] \coloneqq m$; at stage J we can choose m sufficiently large that $n[J] > n[J-1]$. ♦

Proof of (f). The first inequality follows from the fact that the liminf of a product (of non-negative numbers) dominates the product of liminfs; the second is analogous.

By dropping to subsequences we can apply (d) iteratively K times to find an $\vec{n} \prec \vec{s}$ for which

$$\begin{aligned} \mathbf{r}(S_1 \times \cdots \times S_K; \vec{n}) \leq \mathbf{R}(S_1 \times \cdots \times S_K; \vec{n}) &\leq \mathbf{R}(S_1; \vec{n}) \cdot \ldots \cdot \mathbf{R}(S_K; \vec{n}) \\ &= \mathbf{r}(S_1; \vec{n}) \cdot \ldots \cdot \mathbf{r}(S_K; \vec{n}). \end{aligned} \tag{2}$$

This latter is dominated by $\mathbf{r}(S_1 \times \cdots \times S_K; \vec{n})$; hence the above inequalities are equalities. Equality will survive dropping to a subsequence of $\vec{n}$ since all of the (in)equalities of (2) persist. ♦

Calculus gives the following consequence of convexity.

CONVEXITY. *Fix an $r \in [0,1]$ and let E denote the set of K-tuples of real numbers $x_k \in [0,1]$ such that the product $x_1 \cdot x_2 \cdot \ldots \cdot x_K$ equals r. Then the function $f\colon E \to \mathbb{R}$ defined by $f(x_1, \ldots, x_K) \coloneqq \prod_1^K (1 - x_k)$ takes on a maximum at $x_1 = x_2 = \cdots = x_K = \sqrt[K]{r}$. Hence*

$$\big[(1-x_1) \cdot \ldots \cdot (1-x_K)\big]^{1/K} \leq 1 - r^{1/K}$$

for any tuple $(x_1, \ldots, x_K) \in E$.

SPLITTING THEOREM. *If T has a factor which K-fold splits then*

$$[\mathrm{rig}(T)]^{1/K} + [\mathrm{mix}(T)]^{1/K} \leqslant 1.$$

The inequality persists if the rigidity and mixing are computed along any sequence $\vec{s}$.

Remark. Given any number $\rho \in [0,1]$ there is, [2], a weak-mixing map S with $\mathrm{rig}(S) = \rho$ and $\mathrm{rig}(S) + \mathrm{mix}(S) = 1$. Let T be the K-fold cartesian power of S. By computing the effect of T on K-dimensional cubes one sees that $\mathrm{rig}(T) = [\mathrm{rig}(S)]^K$ and $\mathrm{mix}(T) = [\mathrm{mix}(S)]^K$. This shows that the 1 in the righthand side of the theorem cannot be reduced.

Proof. Since partial mixing and rigidity can only increase under passage to factors we may assume T itself splits as $S_1 \times \cdots \times S_K$. Fix a sequence $\vec{s}$. By (e) followed by applying (b) then (d) to $\mathbf{m}$, we may replace $\vec{s}$ by a subsequence and rewrite the desired conclusion as

$$[\mathbf{r}(\vec{s})]^{1/K} + [\mathbf{M}(\vec{s})]^{1/K} \leq 1.$$

Properties (e) and (d) are stable and so for any further subsequence $\vec{n} \prec \vec{s}$ we have $\mathbf{r}(T;\vec{n}) = \mathbf{r}(T;\vec{s})$ and $\mathbf{M}(T;\vec{n}) = \mathbf{M}(T;\vec{s})$. Hence applying (f) to $\mathbf{r}$ and then to $\mathbf{M}$ yields

$$\begin{aligned} \mathbf{r}(T;\vec{n}) &= x_1 \cdot x_2 \cdot \ldots \cdot x_K \\ \mathbf{M}(T;\vec{n}) &\leq (1-x_1)(1-x_2) \cdot \ldots \cdot (1-x_K) \end{aligned}$$

where $x_k \coloneqq \mathbf{r}(S_k;\vec{n})$ and, by (c), $\mathbf{M}(S_k;\vec{n}) \leq 1 - \mathbf{r}(S_k;\vec{n})$. Thus

$$\big[\mathbf{M}(T;\vec{s})\big]^{1/K} \leq 1 - \big[\mathbf{r}(T;\vec{s})\big]^{1/K}$$

by the convexity fact above. ♦

For any non-zero n it is an elementary fact, [3; Prop. 1.13], that $[\mathrm{rig}(T)]^2 \leqslant \mathrm{rig}(T^n) \leqslant \mathrm{rig}(T)$ and $\mathrm{mix}(T^n) = \mathrm{mix}(T)$.

Application. *Given T, pick $K \in \mathbb{N}$ smallest such that*

$$[\mathrm{rig}(T)]^{2/K} + [\mathrm{mix}(T)]^{1/K} \ > \ 1.$$

Then no (non-zero) power of T can K-fold split. So if

$$\mathrm{rig}(T) + \sqrt{\mathrm{mix}(T)} \ > \ 1$$

then no power of T splits.

References

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