Directionless Soldiers Problem: Probability

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
23 September, 2017 (at *16:13*)

Soldiers. N soldiers face the sergeant, who barks "Company, right face!" Alas, the soldiers turn randomly (independently, $\frac{1}{2}, \frac{1}{2}$-probability) left/right. If a soldier finds himself face-to-face with another, he figures that he must have turned wrong; so he reverses (in place) to face the other way (as does the fellow he was facing —now they are back to back).

Assume that the soldiers do this simultaneously, on the count of each second. A soldier who is facing no-one (he faces out, from an end of the line), or who faces someone's back, does *not* turn on this count.

Using notation from below, here is an example with soldiers named $1, 2, \ldots, 5$.

†:

$$\begin{aligned} x &:= \overrightarrow{\mathbf{1}}\,\overleftarrow{2}\,\overleftarrow{3}\,\overrightarrow{\mathbf{4}}\,\overleftarrow{5} \\ x' &= \overleftarrow{1}\,\overrightarrow{\mathbf{2}}\,\overleftarrow{3}\,\overleftarrow{4}\,\overrightarrow{\mathbf{5}} \\ x'' &= \overleftarrow{1}\,\overleftarrow{2}\,\overrightarrow{\mathbf{3}}\,\overleftarrow{4}\,\overrightarrow{\mathbf{5}} \\ \widetilde{x} &= \overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overrightarrow{\mathbf{4}}\,\overrightarrow{\mathbf{5}} \end{aligned}$$

The number of soldier-pairs which reverse is $R(x) = 2 + 1 + 1 = 4$. The number of seconds to stabilize is $S(x) = 3$.

Notation. There are 2^N ***patterns*** of soldiers facing right/left; let $\Omega = \Omega^{\langle N \rangle}$ be this set. With soldiers in pattern x, use x' or $U(x)$ for the successor pattern. We call x a ***predecessor*** of x'. An x with *no* predecessor is called a ***GoE***; a ***garden of Eden*** pattern. Write this set as $\mathrm{GoE}^{\langle N \rangle}$. Use

$$y \rightsquigarrow x$$

to say that pattern y evolves, eventually, to x. So x is GoE iff $[y \rightsquigarrow x \implies y = x]$.

Easily, there are $N+1$ patterns with $x' = x$: At the line's left, there are ℓ soldiers facing left; at the line's right, there are r soldiers facing right; and $\ell + r = N$ are natnums. Let $\perp_r = \perp_r^{\langle N \rangle}$ denote this pattern, and use $\Phi = \Phi^{\langle N \rangle}$ to name this set $\left\{ \perp_r^{\langle N \rangle} \,\middle|\, r \in [0 \,..\, N] \right\}$. We have thus a disjoint union

$$\Omega^{\langle N \rangle} = \bigsqcup_{r \in [0 \,..\, N]} \mathrm{Basin}_r^{\langle N \rangle}.$$

Here, Basin_r is the set of patterns which evolve to $\perp_r$. (The *basin of attraction* of $\perp_r$.)

Ques. *Q1.* Show that every pattern evolves to one in Φ. Let $\widetilde{x}$ denote the final pattern that x becomes. What is the distribution, i.e, what is

$$r \mapsto \mathsf{P}\Big(\mathrm{Basin}_r^{\langle N \rangle}\Big)\,? \qquad \square$$

Q2. As a fnc of N and $r \in [0 \,..\, N]$, what is

$$\text{max-}R_r^{\langle N \rangle} := \operatorname*{Max}_{x \in \mathrm{Basin}_r^{\langle N \rangle}} R(x)\,;$$

and what is a pattern *realizing* this maximum number of reversals? What is its *average value*? :

$$\text{avg-}R_r := \mathsf{E}\Big(R(x) \,\Big|\, x \in \mathrm{Basin}_r\Big). \qquad \square$$

Q3. Analogously to above, what are the maximum and average number of *seconds*, $\text{max-}S_r^{\langle N \rangle}$ and $\text{avg-}S_r^{\langle N \rangle}$, for patterns to stabilize? □

Q4. As $N \to \infty$, where does $\mathsf{P}(\mathrm{GoE}^{\langle N \rangle})$ go? Can we compute $\mathsf{P}(\mathrm{GoE}^{\langle N \rangle})$ exactly? □

Q5. Wrap the line of N soldiers into a circle. Now what are the periodic orbits? What is the basin of attraction of each orbit? How long does it take to fall into a periodic orbit? □

Q6. Now consider a double-infinite row of soldiers, or else a half-infinite ray, and ask all the above questions. □

Analysis

Replace each left-facing soldier by an ***"anchor"*** ◄, and replace each right-facing soldier by a ***"bubble"*** ∘. Pattern x of (†) becomes ∘ ◄◄ ∘ ◄. We can think of a reversal as carrying ∘ ◄ to ◄ ∘; the anchor and bubble interchange positions. By rotating the pattern counter-CW by 90°, we can think of anchors as falling past bubbles that lie immediately below.

Let $\ell + r = N$ denote the respective number of bubbles and anchors in pattern x. We also call $r(x)$ the **"weight"** of x. Starting from the bottom, number the positions of the soldiers as $[0 \,..\, N)$. We can then associate an r-pattern x with the positions $0 \leqslant p_1 < p_2 < \cdots < p_r$. of its anchors. Thus we can view Ω as the set of ordered increasing tuples $\mathbf{p}$ in $[0 \,..\, N)$.

A partial-order on patterns. Define a partial order $\preccurlyeq$ on Ω by: $x \preccurlyeq y$ exactly when $r(x) \leqslant r(y)$ and for each j:

$$p_j(x) \ \leqslant\ p_j(y)\,.$$

Easily $[x \leftsquigarrow y] \Rightarrow [x \preccurlyeq y]$.

Exer. *E1.* $x \preccurlyeq y \implies x' \preccurlyeq y'$. □

Exer. *E2.* For each r, the patterns

$$\top_r := \overbrace{\circ \cdots \circ}^{\ell}\,\overbrace{\blacktriangleleft \cdots \blacktriangleleft}^{r}$$
$$\bot_r = \overbrace{\blacktriangleleft \cdots \blacktriangleleft}^{r}\,\overbrace{\circ \cdots \circ}^{\ell}$$

are the unique $\preccurlyeq$-maximum and $\preccurlyeq$-minimum elements of Basin_r. □

Reversals. Courtesy (E1),

$$\text{max-}R_r^{\langle N\rangle} = R(\top_r^{\langle N\rangle}) = \ell \cdot r\,.$$

Maximizing this over pairs $r + \ell = N$ yields that

$$\text{max-}R^{\langle N\rangle}(\Omega) \ = \ \tfrac{1}{2} \cdot \begin{Bmatrix} N^2 & \text{if } N \text{ even;} \\ N^2 - 1 & \text{if } N \text{ odd.} \end{Bmatrix}.$$

Filename: `Problems/Probability/soldiers.random.latex`
As of: *Sunday 05Oct2003.* Typeset: *23Sep2017* at *16:13.*