The ∞ Snowplow Pile-up

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ABSTRACT: Determining departure times so that ∞ly many snowplows crash simultaneously.

Note. These are problems B and C on P.84, the end of Chap. 2 of *an earlier edition* of Nagle, Saff, Snider.

Notation. Use in, ft, mi for inch(es), foot/feet, mile(s). Use hr and sec for hour(s) and second(s).

Overview. It starts snowing at time-zero. Some time later, snowplows (SPs) start leaving a garage, one after another, plowing a straight road.

The snow falls at accumulation rate $\boldsymbol{\alpha}$, in units in/hr. A SP's speed is inversely proportional to the height of accumulated snow. The multiplier, $\boldsymbol{\mu}$, has units $\frac{\mathsf{mi}}{\mathsf{hr}} \cdot \mathsf{in}$. Were $\boldsymbol{\mu} = 60\,\mathsf{mph} \cdot \mathsf{in}$, for example, then a 4in accumulation would reduce the SP's speed to 15mph.

We'll need ratio $\boldsymbol{\rho} := \frac{\boldsymbol{\alpha}}{\boldsymbol{\mu}}$, which is in 1/mi.

Caveat. When snow accumulation is small, this model says that the speed of a SP is practically infinite. Hmm...

∞-ly many snowplows, Oh My!

Number the snowplows SP_1,SP_2,...,SP_N,.... They leave the garage at times (lag times)

$$0 \cdot \mathsf{hr} < L_1 < L_2 < L_3 < \ldots$$

Let $x_N(\tau)$ be the **position** that SP_N reaches at *time* τ. Let $\tau_N(x)$ be the **time** at which SP_N reaches *position* x.

Observation. It is plausible[♡1] that SP_2 crashes into SP_1 from behind.

Let $\mathbf{X}_*$ denote the position of crash (if there is one) and let $\mathbf{T}_*$ be the time of crash. So $\mathbf{X}_*$ is the smallest positive position such that

1: $$\tau_2(\mathbf{X}_*) = \tau_1(\mathbf{X}_*) =: \mathbf{T}_* .$$

Two Problems. Given lags L_1 and L_2, our job is to compute crash-position $\mathbf{X}_*$ and time $\mathbf{T}_*$.

Then we are to design (figure out) the other lag times $L_3, L_4, \ldots$ so that all the SPs crash *simultaneously* at time $\mathbf{T}_*$.

A solution to ∞ SPs

Consider an $N \geqslant 2$. At position x, the accumulated snow in front of SP_N has height

$$\left[\tau_N(x) - \tau_{N-1}(x)\right] \cdot \boldsymbol{\alpha} .$$

We can make this hold even for the first SP, by defining a fictitious SP_0 having $\tau_0()$ being identically 0 hr. So, for each $N = 1, 2, \ldots,$

$$\frac{\boldsymbol{\mu}}{[\tau_N(x) - \tau_{N-1}(x)] \cdot \boldsymbol{\alpha}} = \text{Speed-of-SP}_N(x) = 1 \Big/ \frac{\mathrm{d}\tau_N}{\mathrm{d}x}(x) .$$

This hands us a differential eqn IVP

2: $$\frac{\mathrm{d}\tau_N}{\mathrm{d}x}(x) = [\tau(x) - \tau_{N-1}(x)] \cdot \boldsymbol{\rho}$$

with initial condition $\boxed{\tau(0\,\mathsf{mi}) = L_N}$. The stage is all set to iteratively solve for fncs $\tau_1, \tau_2, \ldots$.

3: Lemma. *Given constants R and L, and cts fnc $\theta = \theta(x)$, there is a unique differentiable fnc $\tau = \tau(x)$, solving IVP (Initial Value Problem)*

$$\tau' = [\tau - \theta] \cdot R, \quad \textit{with } \tau(0) = L .$$

[♡1] SP_1 is *decelerating*, since the height of snow in front of it is proportional to [its travel time from the garage] $+ L_1$. OTOHand, at any given time, SP_2 sees less accumulated snow than SP_1, so SP_2's speed *always* exceeds SP_1's.

The domain of the soln is all of $\mathbb{R}$. The soln is

$$\tau(x) = Q(x)\cdot \mathsf{e}^{Rx}, \quad \textit{where}$$
$$Q(x) := L - R\cdot\int_0^x \theta(\tilde{x})\cdot \mathsf{e}^{-R\tilde{x}}\cdot \mathrm{d}\tilde{x}\,.$$

Proof. The FOLDE algorithm. ◇

Iteration. In order to conveniently apply this lemma to (2), we define an exponential $E(x) := \mathsf{e}^{\boldsymbol{\rho}\cdot x}$. For $N \geqslant 1$, then,

4: $$\tau_N = Q_N \cdot E\,, \quad \text{where}$$
$$Q_N(x) := L_N - \boldsymbol{\rho}\cdot\int_0^x \tau_{N-1}()\cdot \tfrac{1}{E()}\,.$$

The upper-line gives that $\tau_{N-1}\cdot\frac{1}{E} = Q_{N-1}$. Plugging this into the second line gives the recurrence relation

5: $$Q_N(x) := L_N - \boldsymbol{\rho}\cdot\int_0^x Q_{N-1}()\,.$$

This even holds for $N{=}1$, if we define Q_0 to be identically zero. Recurrence (5) shows that each Q_N is a poly of degree N. The first few are

6: $$\begin{aligned} Q_1(x) &= L_1\,; \\ Q_2(x) &= L_2 - L_1\cdot\boldsymbol{\rho}x\,; \\ Q_3(x) &= L_3 - L_2\cdot\boldsymbol{\rho}x + L_1\tfrac{[\boldsymbol{\rho}x]^2}{2!}\,; \\ Q_4(x) &= L_4 - L_3\cdot\boldsymbol{\rho}x + L_2\tfrac{[\boldsymbol{\rho}x]^2}{2!} - L_1\tfrac{[\boldsymbol{\rho}x]^3}{3!}\,. \end{aligned}$$

The general expressions, valid for all $N \geqslant 0$, are

7: $$\begin{aligned} Q_N(x) &= \sum_{j\in[0\,..\,N)} \frac{[-\boldsymbol{\rho}x]^j}{j!}\cdot L_{N-j}\,; \\ \tau_N(x) &= Q_N(x)\cdot\exp(\boldsymbol{\rho}x)\,. \end{aligned}$$

The second line comes from (4).

Where did the first two SPs collide? We want the positive distance $\mathbf{X}_*$ st. (1). From (4), then, $Q_2(\mathbf{X}_*) = Q_1(\mathbf{X}_*)$. Consequently,

8: $$\mathbf{X}_* = \left[\frac{L_2}{L_1} - 1\right]\cdot\frac{1}{\boldsymbol{\rho}}\,, \quad \text{and so}$$
$$\mathbf{T}_* = \tau_1(\mathbf{X}_*) \overset{\text{note}}{=\!=} L_1\cdot\exp\Bigl(\frac{L_2}{L_1} - 1\Bigr)\,.$$

For future reference, let

$$M := \boldsymbol{\rho}\mathbf{X}_* = \frac{L_2}{L_1} - 1\,.$$

While the *position* of the crash does depend on $\boldsymbol{\rho}$, note that the *time* of crash does not!

What are the lags for a mass-collision? For $N = 3, 4, \ldots,$ we solve for L_N in the equation $Q_N(\mathbf{X}_*) = Q_1(\mathbf{X}_*) \overset{\text{note}}{=\!=} L_1$, to conclude that

9: $$L_N = L_1 - \sum_{j\in[1\,..\,N)} \frac{[1 - \frac{L_2}{L_1}]^j}{j!}\cdot L_{N-j}\,.$$

True or False? For each $K = 1, 2, 3, \ldots$

10: $$L_K \overset{?}{=} L_1\cdot\Bigl[1 + M + \frac{M^2}{2!} + \cdots + \frac{M^{K-1}}{[K{-}1]!}\Bigr]\,.$$

Can you use (9) to show this? □

When did it start snowing?

(Here is a new problem, with just one SP. We do *not* know the values $\boldsymbol{\alpha}$ nor $\boldsymbol{\mu}$. Indeed, the only data that we *do* have are the times

$$D := T_2 + T_1 \quad \text{and} \quad E := T_3 + T_2 + T_1\,,$$

where times $T_{1,2,3}$ are defined below.)

Promptly at 8AM this morning, we cheerfully set out snowplowing. However, we don't know when it started snowing during the night. That is, we started plowing at some unknown lag time –call it λ– after snow-start. Our job is to figure out *when* it started snowing, given the data D and E below.

Let $T_{1,2,3}$ denote the times it took to clear miles $1, 2, 3$, respectively. So $T_3 > T_2 > T_1 > 0\,\mathsf{hr}$. Letting D be the time to clear miles 1&2, we have that $T_2 > \frac{D}{2} > T_1$. Let E the time to clear miles $1, 2$&3. So $T_3 = E - D$. Consequently, $E - D > \frac{D}{2}$. The upshot is this:

11: $$2E - 3D > 0\,,$$ The "positive snowfall-rate condition".

Solution. From the given info,

$$2\mathsf{mi} \;=\; \int_{\lambda}^{\lambda+D} \mathrm{Speed}(t)\,\mathrm{d}t \;=\; \int_{\lambda}^{\lambda+D} \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}\cdot t}\,\mathrm{d}t$$
$$= \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} \cdot \log\Bigl(\frac{\lambda+D}{\lambda}\Bigr).$$

Similarly

$$3\mathsf{mi} \;=\; \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} \cdot \log\Bigl(\frac{\lambda+E}{\lambda}\Bigr).$$

Divide this by the previous, then cross-mult, then exponentiate to obtain

$$\Bigl[\frac{\lambda+E}{\lambda}\Bigr]^2 \;=\; \Bigl[\frac{\lambda+D}{\lambda}\Bigr]^3.$$

Mult. each side by λ^3, then subtract, to discover that λ is a zero of quadratic polynomial

12: $$f(\ell) \;:=\; [2E-3D]\ell^2 + [E^2-3D^2]\ell - D^3\,.$$

Courtesy (11), $\mathrm{Coeff}(\ell^2)$ is positive. And $\mathrm{Coeff}(\ell^0) = -D^3$ is negative. Since these coeffs have opposite signs, our poly f has *positive discriminant*. Thus $f()$ has two real roots, one negative and one positive. Now $\mathrm{Discr}(f)$ equals

$$[E^2-3D^2]^2 \;-\; 4\cdot[2E-3D]\cdot[\text{-}D^3]$$
$$=\; E^4 + 0 - 6E^2D^2 + 8ED^3 - 3D^4\,.$$

Letting S denote the non-negative sqroot of this we have, by the Quadratic Formula, that

$$\lambda \;=\; \tfrac{1}{2[2E-3D]}\cdot\Bigl[S - [E^2-3D^2]\Bigr].$$

It would be nice if there were a simplification of this. Perhaps you can find one!

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