RREF is unique

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
23 September, 2017 (at *14:34*)

1: RREF Uniqueness Thm. *Consider two matrices* A *and* B *in RREF, having the same dimensions and over the same field* $\mathbf{F}$. *If* $\mathsf{A} \overset{r}{\sim} \mathsf{B}$ [*row-equiv.*] *then* $\mathsf{A} = \mathsf{B}$. ◊

Proof. FTSOContradiction,[♡1] suppose $\mathsf{A} \neq \mathsf{B}$.

Let $\boldsymbol{\alpha}(j)$ denote the j^{th} column of A; ditto $\boldsymbol{\beta}(j)$ for B. Take index K smallest st. $\boldsymbol{\alpha}(K) \neq \boldsymbol{\beta}(K)$.

Let P denote the number of pivot columns that A, hence B (Exer: Why?), has to the left of column-K. Write the pivot-positions as

$$(1, c_1), (2, c_2), \ldots, (P, c_P)$$

where, of course, $c_1 < \cdots < c_P$.

$\boxed{\text{Case: Column } \boldsymbol{\alpha}(K) \text{ is a non-pivot column}}$

Then $\boldsymbol{\alpha}(K)$ has form $\begin{bmatrix} x_1 \\ \vdots \\ x_P \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Thus column $\boldsymbol{\alpha}(K)$ is a linear-combination of the pivot-columns to his left, namely

$$\boldsymbol{\alpha}(K) \quad = \quad \sum_{p=1}^{P} x_p \cdot \boldsymbol{\alpha}(c_p)\,.$$

But row-equivalence *preserves* linear relations among columns,[♡2] hence

$$\boldsymbol{\beta}(K) \quad = \quad \sum_{p=1}^{P} x_p \cdot \boldsymbol{\beta}(c_p)\,.$$

But to the left of column-K, matrices A and B agree. For each index $j < K$, consequently, $\boldsymbol{\alpha}(j) = \boldsymbol{\beta}(j)$. In particular, each $\boldsymbol{\alpha}(c_p) = \boldsymbol{\beta}(c_p)$. Thus

$$\boldsymbol{\beta}(K) \quad = \quad \sum_{p=1}^{P} x_p \cdot \boldsymbol{\alpha}(c_p) \quad \stackrel{\text{note}}{=\!=} \quad \boldsymbol{\alpha}(K)\,,$$

contradicting that $\boldsymbol{\alpha}(K)$ is unequal to $\boldsymbol{\beta}(K)$.

$\boxed{\text{Case: Column } \boldsymbol{\alpha}(K) \textit{ is} \text{ a pivot-column}}$ Since the above argument also applies to matrix B, column $\boldsymbol{\beta}(K)$ must itself be a pivot-column (of B, of course).

As $\boldsymbol{\alpha}(K)$ is a pivot-column with P many pivots to its left, necessarily our $\boldsymbol{\alpha}(K)$ equals the transpose of

$$*: \qquad \big[\overbrace{0 \ \ldots \ 0}^{P \text{ many}}\ 1\ 0\ \ldots\ 0\big]\,.$$

But column $\boldsymbol{\beta}(K)$ is also a pivot column, and it also has P many pivots to *its* left. So $\boldsymbol{\beta}(K)$ equals $(*)$, hence equals $\boldsymbol{\alpha}(K)$; again a contradiction. ♦

Filename: `Problems/Algebra/LinearAlg/rref-unique.latex`
As of: *Friday 05Feb2016.* Typeset: *23Sep2017* at *14:34.*

[♡1] Another approach is induction from left-to-right on the columns, making for a more esthetic proof, but slightly longer.

[♡2] That is to say, $\mathrm{LNul}(\mathsf{A}) = \mathrm{LNul}(\mathsf{B})$.