A lower bound on the rank of mixing extensions

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Abstract. If transformation R is a p-point extension of a mixing T then $p \cdot \mathrm{rank}(T)$ lower bounds the rank of R.

If T is a factor of a transformation R what relationship is there between their respective ranks? Certainly $\mathrm{rk}(T) \leqslant \mathrm{rk}(R)$ but, in general, the rank of R need not exceed the rank of T even if the fibers of R over T are infinite (non-atomic). For example, one can make a weak-mixing rank-1 R as a direct product $T \times S$ of non-atomic transformations. This T will be rigid. However, if one assumes that T has *zero* rigidity, in particular if T is mixing, a coding argument can be applied to show

$$\mathrm{rank}(R) \geqslant p \cdot \mathrm{rank}(T)$$

whenever R is any p-point extension of T.

This is a second application of the Fundamental lemma of [1], which is a prerequisite to this note. Below, we briefly recapitulate the notation of that paper.

Terminology. For a measure preserving map T on a Lebesgue probability space $(X, , \mu)$ the ***rigidity*** number, $\rho(T)$, is the largest number in $[0,1]$ such that there exists a sequence $\{k_n\}$ of integers going to infinity for which

$$\liminf_{n\to\infty} \mu(A \cap T^{-k_n} A) \geqslant \rho(T) \cdot \mu(A)$$

for all A. Evidently a mixing T has zero rigidity, $\rho(T) = 0$.

Given a partition P on X which generates under T and given an $x \in X$, let $x|_i$ denote the letter of P in which $T^i x$ finds itself and let $x|_a^b$ denote the substring $x|_a \ x|_{a+1} \cdots x|_{b-1}$ of length $b - a$. Let $|_a^b$ be a synonym for the "interval of integers" $[a, b) \cap \mathbb{Z}$.

Rank and coverings. Suppose $\mathbf{C} = \langle W_1, \ldots, W_r \rangle$ is an ordered set of not-necessarily-distinct words all of a common length h, associated to which is a small positive number written $\bar{d}$-err$(\mathbf{C})$; such a $\mathbf{C}$ was called a ***palette***. Given a name $x \in X$ suppose we have a sequence $\{i_m, c(m)\}_{m=1}^{\infty}$

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where $c(m) \in \{1,\ldots,r\}$ and i_m is a natural number with $i_m + h \leqslant i_{m+1}$. Say that $\{i_m, c(m)\}_{m=1}^{\infty}$ is an ε***-covering*** of $x|_0^\infty$ if

$$\bar{d}\left(x|_{i_m}^{i_m+h}, W_{c(m)}\right) \leqslant \bar{d}\text{-err}(\mathbf{C})$$

and $\bar{d}$-err$(\mathbf{C}) \leqslant \varepsilon$ and the set $\cup_{m=1}^{\infty}[i_m, i_m+h)$ of integers has lower density in $\mathbb{N}$ exceeding $1-\varepsilon$. Let len$(\mathbf{C})$ denote h and $|\mathbf{C}|$ denote r.

Definition 1. The (uniform) ***rank*** of an ergodic T, written rk(T), can be computed from any generating partition P. Define rk(T) to be the smallest r for which: Given any ε and N there exists a palette $\mathbf{C}$ of r words, len$(\mathbf{C}) > N$, which can ε-cover (almost) every name $x|_0^\infty$.

Shift metric. Suppose we have two words $W = W|_0^h$ and $B = B|_0^h$ of common length h. Shift B by n positions and say that the $\bar{d}$ distance between W and the shifted B is

$$\frac{1}{h}\left|\{i \in |_0^h \;:\; W|_i \neq B|_{i+n}\}\right| \tag{2}$$

with the understanding that when $i+n$ does not lie in $|_0^h$ say that $W|_i \neq B|_{i+n}$. Now define the ***shift distance*** $\bar{s}(W,B)$ to be the minimum of (2) taken over all integers n. This $\bar{s}(\cdot,\cdot)$ is a metric on the set of words of length h.

The fundamental lemma. The setting henceforth is an ergodic T, P and two natural numbers r and p. Suppose that for any ε and N there is a palette $\mathbf{C}$, with $|\mathbf{C}| = r$ and len$(\mathbf{C}) > N$, such that any name $x|_0^\infty$ has p **distinct** ε-coverings by the palette. Say that two coverings $\{i_m, c(m)\}_{m=1}^{\infty}$ and $\{i'_k, c'(k)\}_{k=1}^{\infty}$ are ***distinct*** if whenever $i_m = i'_k$ then $c(m) \neq c'(k)$.

The coding argument of [1; §3] obtains following, which we now formulate in more generality.

FUNDAMENTAL LEMMA. *Suppose T has zero rigidity. Given any δ, then for any palette $\mathbf{C} = \langle W_1,\ldots,W_r\rangle$ as above with ε sufficiently small and* len$(\mathbf{C})$ *sufficiently large: Each ball in $\mathbf{C}$ of radius δ in the shift metric contains at least p distinct words.*

That is, for each word $V \in \mathbf{C}$ there are p different values of $c \in \{1,\ldots,r\}$ such that $\bar{s}(V, W_c) < \delta$.

THEOREM. *Suppose T has rigidity zero and for each ε and N there exists a palette $\mathbf{C}$ as above. Then*

$$\text{rk}(T) \leqslant r/p.$$

PROOF. Let t denote rk(T). Evidentally there is a positive δ_0 such that for any palette $\mathbf{C}$ with sufficiently small ε and large length, $\mathbf{C}$ has a δ_0-separated subset with t members; $\bar{s}(V^i, V^j) > \delta_0$ for some subset $V^1,\ldots,V^t \in \mathbf{C}$. This is [1; lemma 7]. Now pick a δ less than $\delta_0/2$ and use the Fundamental lemma to provide a $\mathbf{C}$. The radius δ balls centered at the V^i are disjoint and each contains p members. Thus $r \geqslant p \cdot t$. ♦

Creating a palette C for T. Now suppose $T : X \to X$ is a factor of an ergodic transformation $R : Y \to Y$ with $r := \text{rk}(R)$ finite. By the cutting and stacking characterization (see [2; §1] for details) of rank, one can build a generating partition Q for R such that there exist arbitrarily good palettes $\mathbf{C}$ for the process R, Q having $\bar{d}$-err$(\mathbf{C})$ equal to zero.

Let $\varphi : Y \to X$ denote the factor homomorphism. Given any $\alpha > 0$ there exists a finite code $\widehat{\varphi}$ of codelength, say, L, which approximates φ: The set of Q-words of length $2L$ is mapped by $\widehat{\varphi}$ into the letters of P. Moreover, for (almost) every $y \in Y$

$$\text{density}\big(i \in \mathbb{N} \ : \ \widehat{\varphi}(y|_{i-L}^{i+L}) \neq \varphi(y)|_i\big) \ < \ \alpha.$$

Suppose $\{i_m, c(m)\}$ is a ε-covering with zero $\bar{d}$-error of some name $y|_0^\infty$ by a palette $\mathbf{C}' = \langle W_1', \ldots, W_r' \rangle$ of Q-h-words with length $h \gg L$. Create a palette $\mathbf{C} = \langle W_1, \ldots, W_r \rangle >$ of P-h-words. For each $W' \in \mathbf{C}'$ define the corresponding $W = W|_0^h$ as follows: When i in $|_L^{h-L}$ set $W|_i$ to be $\widehat{\varphi}(W'|_{i-L}^{i+L})$; then define $W|_0^L$ and $W|_{h-L}^h$ in any fixed way.

Evidently $\{i_m, c(m)\}$ is an "ε-covering" of $\varphi(y)|_0^\infty$ by $\mathbf{C}$; **but** with a $\bar{d}$-error of, essentially, $(2L/h) + \alpha$. Having fixed ε, we can pick α and code $\widehat{\varphi}$ and then choose a palette $\mathbf{C}'$ with h sufficiently $\gg L$ so that $(2L/h) + \alpha$ is less than ε. Thus the resulting $\mathbf{C}$ indeed ε-covers $\varphi(y)|_0^\infty$.

Obtaining multiple coverings. Fix an $x \in X$ and then two distinct points $y, z \in \varphi^{-1}(x)$. Palette $\mathbf{C}'$ ε-covers each name: $y|_0^\infty$ by, say, $\{i_m, c(m)\}_{m=1}^\infty$ and $z|_0^\infty$ by $\{j_k, d(k)\}_{k=1}^\infty$. As argued above, these each give rise to a covering of $x|_0^\infty$ by $\mathbf{C}$. Now, without loss of generality these two coverings are distinct: A consequence of the ergodic theorem yields that, when y and z chosen sufficiently generic,

$$\text{UpperDensity}\big\{m : \exists k \text{ with } j_k = i_m \text{ and } d(k) = c(m)\big\}$$

can have been made arbitrarily small by having chosen $\mathbf{C}'$ with $\text{len}(\mathbf{C}')$ sufficiently large. For a proof, see the joining argument of [3; §3].

Finally, suppose that the fibers of R over T have at least p points. Then picking first x and distinct generic points $y_1, \ldots, y_p \in \varphi^{-1}(x)$ we obtain arbitrarily good palettes $\mathbf{C}$ which have p distinct coverings of $x|_0^\infty$. The theorem above yields that $\text{rk}(R) \geqslant p \cdot \text{rk}(T)$.

References

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