Rank-1 has zero entropy

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31 August, 2015 (at *10:09*)

Using stacks

Let $\overrightarrow{\text{STKs}} := (\Xi_n)_{n=1}^{\infty}$ be the stacks used to cut&stack a rank-1 T, on a non-atomic Lebesgue space $(X, \mathcal{X}, \mu)$. Let L_n denote the height of Ξ_n.

Let $\mathsf{S}_n \subset X$ denote the spacers adjoined to make the n-stack. Thus

$$\Xi_n \sqcup \mathsf{S}_{n+1} = \Xi_{n+1} .$$

Let $\mathsf{A}_n := \bigsqcup_{j=n+1}^{\infty} \mathsf{S}_j$ be the spacers adjoined <u>a</u>fter stage-n. Certainly $\mu(\mathsf{A}_n) \searrow 0$.

I first construct a particular 2-set generating partition $\mathsf{P} = \langle\!\langle B, G \rangle\!\rangle$.

Step 1. For $(\Xi_n)_{n=1}^{\infty}$, I will DTASARenumber (Drop To A Subsequence And Renumber) several times, so as to gain a new property for $\overrightarrow{\text{STKs}}$. Later subsequencings will preserve the properties obtained earlier. I can initially DTASARenumber so that $L_n > n+3$, for all n.

First DTASARenumber so that there are at least 2^n copies of Ξ_n in Ξ_{n+1}. Now simply declare that the bottom-most copy of Ξ_n in Ξ_{n+1} is, in fact, spacer which is part of S_{n+1}. Since $n \mapsto 1/2_n$ is summable, Misters Borel and Cantelli tell me, for (almost) every x, that I changed my mind only finitely often as to what stage the point x was adjoined.

Courtesy of the above, then,

1: *At least the bottom $n+2$ many levels of Ξ_n are in S_n, that is, are n-spacer.*,

Step 2. Fix a sequence $\varepsilon_n \searrow 0$ and arrange, by DTASARenumbering, that

$$*: \qquad \forall n: \quad L_n \cdot \mu(\mathsf{A}_{n+1}) < \varepsilon_n .$$

How? Well, let $k_1 := 1$. Having already chosen $k_1 < k_2 < \cdots < k_n$, pick $\ell > k_n$ smallest so that $L_{k_n} \cdot \mu(\mathsf{A}_\ell) < \varepsilon_n$. Now let $k_{n+1} := \ell$. Continue. Lastly, renumber by renaming Ξ_{k_n} to Ξ_n. Now $(*)$ holds.

I can drop again to arrange that each ratio L_n/L_{n+1} be as small as desired. So it can certainly be arranged that

$$2: \qquad L_n \cdot \Big[\mu(\mathsf{A}_{n+1}) + \frac{L_n}{L_{n+1}}\Big] < \varepsilon_n ,$$

for each n.

Defining a partition. Define $\mathsf{P} = \langle\!\langle B, G \rangle\!\rangle$, by painting Blue and Green levels as follows: For $n = 1, 2, \ldots,$ the bottom $[n+2]$ levels of Ξ_n are all in S_n. Paint one level BLUE, then n levels GREEN, followed by one BLUE. This is the n-***tag***. Lastly, paint BLUE the *remaining* levels in S_n.

Courtesy of (**??**), only spacer is being painted so there is no erasing going on. Since P allows one to unambiguously recognize the base of Ξ_n, it is a generator for T.

Counting L-words entirely within Ξ_{n+1}.

Fix n and let $L := L_n$. How many T,P-h-words W are there, that start and stay in the $[n+1]$-stack?

There are $n+3$ many L-words starting from the $n+3$ bottom-most levels of Ξ_{n+1}. Since the $[n+1]$-tag is here, this is the only part of S_{n+1} which is not all BLUE.

Starting elsewhere in Ξ_{n+1}, an L-word is determined by three non-negative numbers $t+s+b = L$. Our L-word sees the *top* t levels of Ξ_n (some colored BLUE, some GREEN), followed by s levels of $[n+1]$-spacer (all BLUE), followed by the *bottom* b many levels of Ξ_n. (Any of these three numbers could be zero.) Since W is determined by these three numbers, there are at most L^2 many words W. Together with the first paragraph, there are at most

$$n+3+L^2 \overset{\text{WLOG}}{<} L^3$$

many P-words which start and stay in the $[n+1]$-stack.

Upper bounding distropy

If an L-word W wasn't counted above, then it must start either in A_{n+1}, or within the top L levels of Ξ_{n+1}. Let E denote the union of these two sets, and use $\mathbf{E}$ for the ptn (E, E^c). Let $\delta := \mu(E)$ and $\delta^c := 1 - \delta$.

Note that $\mathsf{P}_0^{L-1} \preccurlyeq \mathbf{E} \vee \mathsf{P}_0^{L-1}$. Thus

$$\begin{aligned}\mathcal{H}(\mathsf{P}_0^{L-1}) \leqslant\ & \delta \cdot [\text{Max distropy of a ptn with } 2^L \text{ atoms}] \\ & \delta^c \cdot [\text{Max distropy of a ptn with } L^3 \text{ atoms}] \\ \leqslant\ & \delta L + \delta^c \cdot 3\log(L)\end{aligned}$$

This, together with (**??**) gives

3: $$\tfrac{1}{L_n}\mathcal{H}(\mathsf{P}_0^{L_n-1}) \ \leqslant\ \varepsilon_n L_n + 3 \cdot \tfrac{\log(L_n)}{L_n}$$

4: Theorem. *If T is rank-1 then $\mathcal{E}(T)$ is zero.* ◊

Proof. Take a two-set generator P, as above. By class theorems,

$$\mathcal{E}(T, \mathsf{P}) \ = \lim_{n\to\infty} \tfrac{1}{L_n}\mathcal{H}(\mathsf{P}_0^{L_n-1})\,.$$

And by (**??**), this is zero. Hence Bob is our Uncle. ♦

Filename: `Problems/Dynamics/RankOne/rank1-zeroentr.latex`
As of: *Monday 31Aug2015.* Typeset: *31Aug2015* at *10:09.*