

## Rank + Nullity theorem

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**Entrance.** Here,  $\mathbf{V}, \mathbf{H}$  are finite-dimensional VSpaces over the same field, and  $\mathsf{T}: \mathbf{V} \rightarrow \mathbf{H}$  is linear. Recall

$$\begin{aligned} \text{Nullity}(\mathsf{T}) &:= \text{Dim}(\text{Nul}(\mathsf{T})), \quad \text{and} \\ \text{Rank}(\mathsf{T}) &:= \text{Dim}(\text{Range}(\mathsf{T})). \end{aligned}$$

**1: Rank-Nullity Thm.** For  $\mathsf{T}: \mathbf{V} \rightarrow \mathbf{H}$  as above,

$$\begin{aligned} \text{Rank}(\mathsf{T}) + \text{Nullity}(\mathsf{T}) &= \text{Dim}(\text{Dom}(\mathsf{T})) \\ &\stackrel{\text{here}}{=} \text{Dim}(\mathbf{V}). \quad \diamond \end{aligned}$$

**Proof.** Choose a basis  $\mathcal{Z} = \{\mathbf{z}_1, \dots, \mathbf{z}_N\}$  for  $\text{Ker}(\mathsf{T})$ . [Mnemonic:  $\mathcal{Z}$  is for “zero”.] Pick a basis

$$\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_R\}$$

for  $\text{Range}(\mathsf{T})$ . Finally, for each index  $r \in [1..R]$  pick a vector  $\widehat{\mathbf{h}}_r \in \mathsf{T}^{-1}(\mathbf{h}_r)$ ; possible, since each  $\mathbf{h}_r$  is in  $\text{Range}(\mathsf{T})$ .

Our goal is to show that list

$$\mathcal{B} := (\mathbf{z}_1, \dots, \mathbf{z}_N, \widehat{\mathbf{h}}_1, \dots, \widehat{\mathbf{h}}_R)$$

is a  $\mathbf{V}$ -basis.

**Proof:  $\mathcal{B}$  is L.I.** Consider a linear-combination

$$*: \quad \left[ \sum_{k=1}^N \beta_k \mathbf{z}_k \right] + \left[ \sum_{r=1}^R \alpha_r \widehat{\mathbf{h}}_r \right] = \mathbf{0}_{\mathbf{V}}.$$

Applying  $\mathsf{T}()$  yields that

$$***: \quad \mathbf{0}_{\mathbf{H}} + \left[ \sum_{r=1}^R \alpha_r \mathbf{h}_r \right] = \mathbf{0}_{\mathbf{H}},$$

since each  $\mathsf{T}(\mathbf{z}_k) = \mathbf{0}_{\mathbf{H}}$ , and  $\mathsf{T}(\widehat{\mathbf{h}}_r) = \mathbf{h}_r$ . But  $\mathcal{H}$  is L.I.; consequently all the  $\alpha$ -numbers are zero.

Our (\*) now says that

$$\sum_{k=1}^N \beta_k \mathbf{z}_k = \mathbf{0}_{\mathbf{V}}.$$

But  $\mathcal{Z}$  is L.I., so all  $\beta$ -numbers are zero. QED

**Pf: Subspace  $\text{Spn}(\mathcal{B})$  is all of  $\mathbf{V}$ .** Consider an arbitrary vector  $\mathbf{p} \in \mathbf{V}$ . Thus there exist scalars  $\alpha_r$  st.

$$\mathsf{T}(\mathbf{p}) = \sum_{r=1}^R \alpha_r \mathbf{h}_r.$$

Define

$$\widehat{\mathbf{p}} := \sum_{r=1}^R \alpha_r \widehat{\mathbf{h}}_r.$$

Since  $\mathsf{T}$  is linear,

$$\mathsf{T}(\mathbf{p} - \widehat{\mathbf{p}}) = \mathsf{T}(\mathbf{p}) - \mathsf{T}(\widehat{\mathbf{p}}) \stackrel{\text{note}}{=} \mathbf{0}_{\mathbf{H}}.$$

Thus difference-vector  $\mathbf{p} - \widehat{\mathbf{p}} \in \text{Ker}(\mathsf{T})$ . So there exist scalars  $\beta_k$  for which

$$\sum_{k=1}^N \beta_k \mathbf{z}_k = \mathbf{p} - \widehat{\mathbf{p}}.$$

Hence vector

$$\mathbf{p} \stackrel{\text{note}}{=} \left[ \sum_{k=1}^N \beta_k \mathbf{z}_k \right] + \left[ \sum_{r=1}^R \alpha_r \widehat{\mathbf{h}}_r \right] \quad \blacklozenge$$

is in  $\text{Spn}(\mathcal{B})$ . QED

*The next page has the same proof, but allowing  $\infty$ -dim'al spaces. It indexes vectors by them selves, hence is notationally simpler.*

**Prolegomenon.** Here,  $\mathbf{V}, \mathbf{H}$  are VSes over a field  $\mathbf{F}$ , and  $T: \mathbf{V} \rightarrow \mathbf{H}$  is linear. Recall

$$\begin{aligned} \text{Nullity}(T) &:= \text{Dim}(\text{Nul}(T)), \quad \text{and} \\ \text{Rank}(T) &:= \text{Dim}(\text{Range}(T)), \end{aligned}$$

where these dimensions are (possibly infinite) cardinals.

Recall that a **linear-combination** over a set,  $\Omega$ , of vectors, has form

$$\sum_{\omega \in \Omega} \varphi(\omega) \cdot \omega,$$

where function  $\varphi: \Omega \rightarrow \mathbf{F}$  is **finitely supported**; that is, the set  $\{\omega \in \Omega \mid \varphi(\omega) \neq 0\}$  is finite.

**2: General Rank-Nullity Thm.** For  $T: \mathbf{V} \rightarrow \mathbf{H}$  as above,

$$\text{Rank}(T) + \text{Nullity}(T) = \text{Dim}(\text{Dom}(T)). \quad \diamond$$

**Proof.** Pick a basis  $\mathcal{Z}$  for  $\text{Ker}(T)$ , and a basis  $\mathcal{H}$  for  $\text{Range}(T)$ . For each  $\mathbf{h} \in \mathcal{H}$ , choose a vector  $\widehat{\mathbf{h}} \in T^{-1}(\mathbf{h})$ . Define

$$\widehat{\mathcal{H}} := \{\widehat{\mathbf{h}} \mid \mathbf{h} \in \mathcal{H}\}.$$

Our goal: The disjoint union  $\mathcal{B} := \mathcal{Z} \cup \widehat{\mathcal{H}}$  [of multisets] is a  $\mathbf{V}$ -basis.

**Pf:  $\mathcal{B}$  is L.I.** Consider a linear-combination

$$*: \quad \left[ \sum_{\mathbf{z} \in \mathcal{Z}} \beta(\mathbf{z}) \cdot \mathbf{z} \right] + \left[ \sum_{\mathbf{h} \in \widehat{\mathcal{H}}} \alpha(\mathbf{h}) \cdot \widehat{\mathbf{h}} \right] = \mathbf{0}_{\mathbf{V}}.$$

Applying  $T()$  yields that

$$**: \quad \mathbf{0}_{\mathbf{H}} + \left[ \sum_{\mathbf{h} \in \widehat{\mathcal{H}}} \alpha(\mathbf{h}) \cdot \mathbf{h} \right] = \mathbf{0}_{\mathbf{H}},$$

since each  $T(\widehat{\mathbf{h}}) = \mathbf{h}$ . But  $\widehat{\mathcal{H}}$  is L.I; consequently the  $\alpha()$  function is identically-zero.

Our (\*) now says that

$$\left[ \sum_{\mathbf{z} \in \mathcal{Z}} \beta(\mathbf{z}) \cdot \mathbf{z} \right] = \mathbf{0}_{\mathbf{V}}.$$

But  $\mathcal{Z}$  is L.I, so  $\beta()$  is identically-zero. QED

**Pf: Subspace  $\text{Spn}(\mathcal{B})$  is all of  $\mathbf{V}$ .** Consider an arbitrary vector  $\mathbf{p} \in \mathbf{V}$ . Thus there exists  $\alpha()$  so that

$$T(\mathbf{p}) = \sum_{\mathbf{h} \in \mathcal{H}} \alpha(\mathbf{h}) \cdot \mathbf{h}.$$

Define

$$\widehat{\mathbf{p}} := \sum_{\mathbf{h} \in \mathcal{H}} \alpha(\mathbf{h}) \cdot \widehat{\mathbf{h}}.$$

Because  $T$  is linear,

$$T(\mathbf{p} - \widehat{\mathbf{p}}) = T(\mathbf{p}) - T(\widehat{\mathbf{p}}) \stackrel{\text{note}}{=} \mathbf{0}_{\mathbf{H}}.$$

Since difference-vector  $\mathbf{p} - \widehat{\mathbf{p}} \in \text{Ker}(T)$ , there exists a linear-combination  $\left[ \sum_{\mathbf{z} \in \mathcal{Z}} \beta(\mathbf{z}) \cdot \mathbf{z} \right]$  that equals difference  $[\mathbf{p} - \widehat{\mathbf{p}}]$ . Hence

$$\mathbf{p} \stackrel{\text{note}}{=} \left[ \sum_{\mathbf{z} \in \mathcal{Z}} \beta(\mathbf{z}) \cdot \mathbf{z} \right] + \left[ \sum_{\mathbf{h} \in \mathcal{H}} \alpha(\mathbf{h}) \cdot \widehat{\mathbf{h}} \right] \quad \blacklozenge$$

is in  $\text{Spn}(\mathcal{B})$ . QED

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