Ramsey-like theorems

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24 April, 2018 (at *10:21*)

ABSTRACT: Ramsey-like theorems. Hales-Jewett.

In this note, a ***coloring*** *is an edge-coloring unless specified otherwise.*

Entrance. (jk: Jeudi, 26 mars, 1985) Let K_n denote the *complete graph* on n vertices. Let $\mathcal{R}(n)$ denote the n^{th} ***Ramsey number*** , that is, the smallest M such that if one colors each edge of K_M either blue or green, then there must exist a monochromatic subgraph isomorphic to K_n.

Notation. For vertex $\mathbf{w}$ in a blue-green graph, let $bV(\mathbf{w})$ be the *set* of vertices connected to $\mathbf{w}$ by a blue edge; use $gV(\mathbf{w})$ for those connected to $\mathbf{w}$ by green edges. We thus have vertex degrees

$$\mathrm{bDeg}(\mathbf{w}) := |bV(\mathbf{w})| \quad \textit{and} \quad \mathrm{gDeg}(\mathbf{w}) := |gV(\mathbf{w})|.$$

Hence $\mathrm{bDeg}(\mathbf{w}) + \mathrm{gDeg}(\mathbf{w}) = \mathrm{Deg}(\mathbf{w})$.

1a: Theorem. *For* $n = 1, 2, 3, \ldots,$

$$1a': \qquad \mathcal{R}(n+1) \leqslant 2\big[1 + [n-1]\mathcal{R}(n)\big]. \qquad \diamondsuit$$

Proof. With $M := \mathrm{RhS}(1a')$, let $\mathbb{V}$ denote the vertex set of K_M. FTSOC, assume we have edge-colored K_M so that it has *no* monochromatic copy of K_{n+1}.

Pick a $\mathbf{u}_0 \in \mathbb{V}$. WLOGenerality at least half of the edges from $\mathbf{u}_0$ are blue. Letting $B_0 := \mathrm{bDeg}(\mathbf{u}_0)$, we have

$$*: \qquad B_0 \geqslant \left\lceil \frac{M-1}{2} \right\rceil \stackrel{\text{note}}{=\!=} 1 + [n-1] \cdot \mathcal{R}(n).$$

Building a blue complete-subgraph. Pick a vertex $\mathbf{u}_1 \in bV(\mathbf{u}_0)$. The set $bV(\mathbf{u}_0) \setminus \{\mathbf{u}_1\}$ can have at most $\mathcal{R}(n) - 1$ vertices not belonging to $bV(\mathbf{u}_1)$. If otherwise, then there would be a copy of $K_{\mathcal{R}(n)}$ in our graph, disjoint from $\mathbf{u}_0$ and $\mathbf{u}_1$, such that each vertex in this copy was connected to $\mathbf{u}_0$ by a blue edge and to $\mathbf{u}_1$ by a green edge. This copy would, by hypothesis, contain a monochromatic copy –call it H– of K_n. Were H blue, then $H \sqcup \{\mathbf{u}_0\}$ is a blue K_{n+1}. Else, H is green, so $H \sqcup \{\mathbf{u}_1\}$ is a green K_{n+1}. Either is a ※.

Consequently the set $bV(\mathbf{u}_0) \cap bV(\mathbf{u}_1)$ has at least

$$|bV(\mathbf{u}_0)| - |\{\mathbf{u}_1\}| - [\mathcal{R}(n) - 1] \stackrel{\text{note}}{=\!=} B_0 - \mathcal{R}(n)$$

many vertices. Pick some $\mathbf{u}_2$ in $bV(\mathbf{u}_0) \cap bV(\mathbf{u}_1)$. The same argument shows $bV(\mathbf{u}_0) \cap bV(\mathbf{u}_1) \cap bV(\mathbf{u}_2)$ has at least

$$[B_0 - \mathcal{R}(n)] - \mathcal{R}(n)$$

many vertices.

Continue. At stage $n-1$ we will have chosen distinct vertices $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$. Further, the intersection of their blue vertex-sets will satisfy

$$\Big| bV(\mathbf{u}_0) \cap bV(\mathbf{u}_1) \cap \ldots \cap bV(\mathbf{u}_{n-1}) \Big| \geqslant B_0 - [n-1]\mathcal{R}(n)$$

and hence be non-empty, by ($*$). Picking a vertex $\mathbf{u}_n$ in this intersection, now every pair of vertices in $\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is connected by a blue edge. ※ ♦

Remark. Values $\mathcal{R}(1)$, $\mathcal{R}(2)$ and $\mathcal{R}(3)$ are 1, 2 and 6 respectively. Using the above theorem, easily

$$\mathcal{R}(n+1) \leqslant [n! \cdot 2^n]. \qquad \square$$

More colors.

If we allow $\mu \in \mathbb{Z}_+$ many colors, then we get a corresponding sequence of Ramsey numbers $\mathcal{R}_\mu(n)$ for $n = 1, 2, \ldots$. Here, $\mathcal{R}_1(n) = n$, and $\mathcal{R}_2(\cdot)$ is another name for $\mathcal{R}(\cdot)$.

For each $\mu{>}1$, lump the μ colors into two ***supercolors*** consisting each of at most $h := \lceil \mu/2 \rceil$ colors. Applying $\mathcal{R}(\cdot)$ to these two supercolors, implies that

$$\mathcal{R}_\mu(n) \leqslant \mathcal{R}(\mathcal{R}_h(n)) \leqslant \mathcal{R}(\mathcal{R}(\ldots\mathcal{R}(n)\ldots)),$$

where the RhS has $\lceil \log_2(\mu) \rceil$ occurrences of $\mathcal{R}(\cdot)$. $\square$

Infinite Ramsey's Theorem

(jk, Feb1989) Let K_∞ denote the complete graph on denumerably many vertices.

1b: Infinite Ramsey's Theorem. *Each edge $\overline{\mathbf{u}\,\mathbf{v}}$ of $H = (\mathbb{V}, \mathbb{E})$, a K_∞ graph, is colored either blue or green. Then H includes a monochromatic K_∞.* ♢

Proof. Let $\mathrm{W}_0 := \mathbb{V}$. At stage N we have vertex-sets

$$\mathrm{W}_0 \supset \mathrm{W}_1 \supset \cdots \supset \mathrm{W}_j \supset \cdots \supset \mathrm{W}_N\,,$$

each infinite. Moreover, we have vertices $(\mathbf{u}_j)_{j=0}^{N-1}$, with $\mathbf{u}_j \in \mathrm{W}_j \smallsetminus \mathrm{W}_{j+1}$, so that each edge-set

$$\mathbb{E}_j := \{\overline{\mathbf{v}\,\mathbf{u}_j} \mid \mathbf{v} \in \mathrm{W}_{j+1}\}$$

is monochromatic.

Continue the induction by picking an arbitrary vertex $\mathbf{u}_N \in \mathrm{W}_N$. Since $\mathrm{W}_N \smallsetminus \{\mathbf{u}_N\}$ is infinite, it includes an infinite set of vertices $\mathbf{v}$ so that $\{\overline{\mathbf{v}\,\mathbf{u}_N}\}_{\mathbf{v}}$ is monochromatic. Define W_{N+1} to be this infinite set of vertices.

An infinite monochromatic sequence. Each of the sets $\mathbb{E}_0, \mathbb{E}_1, \ldots$ is either blue or green. Thus there is a subsequence $N_1 < N_2 < \ldots$ so that, WLOG, each $\mathbb{E}_{N_j}$ is green. The consequence is that the complete graph with vertex-set $(\mathbf{u}_{N_j})_{j=1}^{\infty}$ has every edge green.♦

1c: Observation. *Infinite Ramsey's Thm implies the Finite Ramsey's Thm, by a compactness argument.* ♢

Proof. Consider K_∞, the complete graph with vertex-set $\mathbb{V} = \{1, 2, \ldots\}$.

FTSOC suppose the, say, 5^{th} Ramsey number were not finite. Then for each n, there would exist a blue-green coloring of the edges on vertex-set $\{1, \ldots, n\}$ –call this colored graph H_n– so that H_n does not include a monochromatic copy of K_5.

Interpret H_n as an edge-coloring of K_∞: *Color plum each edge that is not between vertices* $[1\,..\,n]$. (I.e, for posints $\mathbf{u} < \mathbf{v}$: If $\mathbf{v} > n$ then color edge $\overline{\mathbf{u}\,\mathbf{v}}$ plum.)

We now have a sequence $H_1, H_2, H_3, \ldots$ of $\{$*blue*, *green*, *plum*$\}$ colorings of K_∞. Since each edge of K_∞ has only finitely many [three] possible colors, the sequence of colorings has a convergent subsequence $(H_{n_j})_{j=1}^{\infty}$. The coloring obtained in the limit, an edge-coloring of K_∞, has *no* plum. But ∞-Ramsey's-Thm guarantees a monochromatic copy of K_∞; WLOG it is green. Letting $\mathbf{u}_1 < \mathbf{u}_2 < \ldots < \mathbf{u}_5$ be its first 5 vertices, this green K_5 was *already* a subgraph of $H_{\mathbf{u}_5}$. ※

2a: *Two-variable $\mathcal{R}()$.* For posints b,g, let $\mathcal{R}(\mathsf{b}, \mathsf{g})$ be the smallest M st. each blue-green-coloring of K_M has either a blue K_{b} or a green K_{g}. So $\mathcal{R}(n) = \mathcal{R}(n, n)$.

Evidently, $\mathcal{R}(\mathsf{b}, 1) = 1$ and $\mathcal{R}(\mathsf{b}, 2) = \mathsf{b}$; and this holds symmetrically, since $\mathcal{R}(\mathsf{b}, \mathsf{g}) = \mathcal{R}(\mathsf{g}, \mathsf{b})$. □

2b: Theorem. *For all* $\mathsf{b}, \mathsf{g} \in \mathbb{Z}_+$,

$$*: \qquad \mathcal{R}(\mathsf{b}, \mathsf{g}) \leqslant \mathcal{R}(\mathsf{b}{-}1, \mathsf{g}) + \mathcal{R}(\mathsf{b}, \mathsf{g}{-}1)\,. \qquad ♢$$

Proof. Let $M :=$ RhS$(*)$. Fix a vertex $\mathbf{w}$; it has $M{-}1$ edges, so has either at least $\mathcal{R}(\mathsf{b}{-}1, \mathsf{g})$ blue-edges, *or* at least $\mathcal{R}(\mathsf{b}, \mathsf{g}{-}1)$ green-edges. WLOG

$$\big|bV(\mathbf{w})\big| \geqslant \mathcal{R}(\mathsf{b}{-}1, \mathsf{g}).$$ [*Recall $bV(\mathbf{w})$ is the set of vertices blue-connected to $\mathbf{w}$.*]

If the graph induced by $bV(\mathbf{w})$ has a green K_{g}, then DONE. Else, the induced graph admits a blue $K_{\mathsf{b}-1}$ which, together with vertex $\mathbf{w}$, induces a blue K_{b}. ♦

2c: *Exercise:* Coloring with μ colors, let $\mathcal{R}(n_1, \ldots, n_\mu)$ be the smallest M st. each μ-coloring, H, of K_M has an index j for which H admits a K_{n_j} with all edges the j^{th}-color. *Exer*-E1: What is the μ-color analog of Thm 2b? [*Hint:* Don't jump to conclusions.] □

Examples from Bona's text

In class we proved $\mathcal{R}(3)=6$. In this instance (2b$*$) is sharp, as

$$\mathcal{R}(3,3) \leqslant 2\cdot\mathcal{R}(3,2) = 2\cdot 3 = 6\,.$$

Using Thm 2b again,

$$\mathcal{R}(3,4) \leqslant \mathcal{R}(2,4)+\mathcal{R}(3,3) = 4+6 = 10\,.$$

3a: Claim: $\mathcal{R}(3,4)=9$. ♢

Pf of $\mathcal{R}(3,4)\leqslant 9$. FTSOC, suppose H is a blue-green-coloring of K_9 with no blue triangle, nor green K_4.

Could *some* vertex $\mathbf{w}$ have $\mathrm{bDeg}(\mathbf{w}) \geqslant 4$? Since no blue triangle, no pair of vertices in $bV(\mathbf{w})$ has a blue-edge; but then $bV(\mathbf{w})$ induces a green K_4. ※

Could *every* $\mathbf{w}$ have $\mathrm{bDeg}(\mathbf{w})=3$? Then the blue-degree-sum is $3\cdot 9$, which is odd. But this degree-sum must also equal twice the number of blue edges. ※

So *there exists* a vertex $\mathbf{w}$ with $\mathrm{bDeg}(\mathbf{w})\leqslant 2$, hence

$$|gV(\mathbf{w})| \geqslant 8-2 = 6\,.$$

Since $\mathcal{R}(3)=6$, and $gV(\mathbf{w})$ cannot have a blue triangle, it must have a green triangle, G. But then $G\sqcup\{\mathbf{w}\}$ is a green K_4. ♦

Pf of $\mathcal{R}(3,4)>8$. On vertex-set $\mathbb{V} := [0\,..\,8)$, let $\equiv$ and $\oplus$ and $\ominus$ each operate mod 8.

Connect vertices $j,k\in\mathbb{V}$ by blue IFF $k\ominus j$ is either

- *3*, *producing a blue octagon*, **or**
- *4*, *producing four center-crossing blue edges.* ♦

But no triple of numbers from $\{3,4\}$ has $\oplus$-sum equal to zero. Hence there is *no* blue triangle.

Color green the remaining edges, i.e $k\ominus j\in\{1,2\}$. Difference=*2* makes green square $\{0,2,4,6\}$, and square $\{1,3,5,7\}$. And difference=*1* make a green octagon $\{0,1,2,3,4,5,6,7\}$.

FTSOContradiction, suppose G is a green K_4 subgraph. It can't have three vertices in one square [diff=*4*], so G has *two* vertices in *each* square, say $\{0,2\}$ and $\{\mathbf{v},\mathbf{w}\}$. But $\mathbf{v}$ must differ from each of 0,2 by *1* or *2*; so $\mathbf{v}\overset{\text{must}}{=\!=}1$. Ditto, $\mathbf{w}=1$. ※

The result now allows, courtesy Thm 2b, that

$$\mathcal{R}(4,4) \leqslant 2\cdot\mathcal{R}(3,4) = 18\,.$$

3b: Claim: $\mathcal{R}(4,4)=18$. ♢

Proof. To show $\mathcal{R}(4,4)>17$, we exhibit a blue-green-coloring of K_{17} with no monochromatic K_4.

On vertex-set $\mathbb{V} := \{0,1,\ldots,8,\text{-}8,\text{-}7,\ldots,\text{-}1\}$, let $\equiv$, $\oplus$, $\ominus$ and $\langle\cdot\rangle$ each operate mod 17. Blue-connect $j,k\in\mathbb{V}$ IFF $k\ominus j$ is a mod-17 QR [Quadratic Residue].

3c:

x	$\langle x^2\rangle$	x	$\langle x^2\rangle$
±1	1	±5	8
±2	4	±6	2
±3	-8	±7	-2
±4	-1	±8	-4

Thus QR $=\{\pm1,\pm2,\pm4,\pm8\}$; so our edge-lengths are $\{1,2,4,8\}$. And NQR $=\{\pm3,\pm5,\pm6,\pm7\}$, giving rise to the green edges. Multiplying the vertices by a non-QR element, will exchange the blue and green edges [using that mult distrbutes-over addition]. So: *It suffices to show that there is no blue K_4.*

Number Thy. FTSOC, suppose $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{V}$ are the vertices of a blue K_4. WLOG $\mathbf{a}$=0; replace vertex x by $x\ominus\mathbf{a}$. WLOG $\mathbf{b}$=1; replace each x by $\langle x/\mathbf{b}\rangle$. [*The mapping $x\mapsto\langle x/\mathbf{b}\rangle$ is color-preserving, since* $\mathbf{b}$ *is a* QR.]

The vertices are now $\{0,1,\mathbf{c},\mathbf{d}\}$.

Vertex $\mathbf{c}$ must differ from 0 and 1 by values in $\{1,2,4,8\}$; so $\mathbf{c}$ is 2 or -1; WLOG $\mathbf{c}$=2. Vertex $\mathbf{d}$, relative to vertices 1 and 2, must be 3 or 0; but 0 is taken. Our supposedly-blue K_4 is $\{0,1,2,3\}$.

Alas, edge $\overline{03}$ is green. ※ ♦

van der Waerden's thm

A ***shift*** $\mathsf{s} \in \mathbb{Z}$ and a ***gap*** $\mathsf{g} \in \mathbb{Z}_+$, give rise to *finite* and *infinite* ***arithmetic progressions***:

$$\mathrm{AP}_n(\mathsf{s},\mathsf{g}) \coloneqq \{\mathsf{s}+j\mathsf{g} \mid j \in [0\,..\,n)\}\,;$$
$$\mathrm{AP}(\mathsf{s},\mathsf{g}) = \mathrm{AP}_\infty(\mathsf{s},\mathsf{g}) \coloneqq \{\mathsf{s}+j\mathsf{g} \mid j \in \mathbb{Z}\}\,.$$

Use AP_n to refer to *some* n-term A.P, when the shift and gap are irrelevant.

4a: van der Waerden theorem. *For each posint n, there is a finite M so that each 2-coloring (of the elements) of AP_M, produces a monochromatic AP_n.* ♢

The smallest such M is written $\mathcal{W}(n)$. For example, $\mathcal{W}(3) = 9$. If μ many colors are allowed, then we write $\mathcal{W}_\mu(n)$ for the minimum M.

The infinite analog of vdW thm fails, as shown next.

4b: Obs. *There exists a 2-coloring of $\mathbb{Z}$ which admits no monochromatic sub-AP_∞.* ♢

Proof. *We produce a coloring for which the only monochromatic arithmetic-progs are finite.*

Take a surjection $n \mapsto (\mathsf{s}_n, \mathsf{g}_n)$ from $\mathbb{Z}_+ \twoheadrightarrow \mathbb{Z}\times\mathbb{Z}_+$, and let $\mathcal{A}(n) \coloneqq \mathrm{AP}_\infty(\mathsf{s}_n, \mathsf{g}_n)$.

We color $\mathbb{Z}$ in stages. At stage n: Paint one blue and one green, the two posints in $\mathcal{A}(n)$ which are closest to zero, and were not yet colored. Then paint one blue and one green, the two negints in $\mathcal{A}(n)$ which are closest to zero, and were not yet colored.

Send $n \nearrow \infty$. Finally, color arbitrarily the remaining uncolored integers. ♦

Hypergraphs

Let $\mathcal{P}_4(\mathbb{V})$ be the collection of cardinality-4 subsets of $\mathbb{V}$; so it has $\binom{|\mathbb{V}|}{4}$ many members. A "**4-*hypergraph*** on $\mathbb{V}$" is a pair $H := (\mathbb{V}, \mathbb{E})$ where $\mathbb{E} \subset \mathcal{P}_4(\mathbb{V})$ is the set of ***hyper-edges*** of H. This cardinality, 4, is called the ***rank*** of H.

A subset $V' \subset \mathbb{V}$ determines a sub-hypergraph $H' := (V', \mathbb{E}')$ of H, where

$$¥: \qquad \mathbb{E}' := \{S \in \mathbb{E} \mid S \subset V'\}.$$

By the way , the "*complete* 4-hypergraph on $\mathbb{V}$" is $(\mathbb{V}, \mathcal{P}_4(\mathbb{V}))$. Setting $M := |\mathbb{V}|$, we'll use $K_M^{\langle 4\rangle}$ to refer to this hypergraph. [*So* $K_M^{\langle 2\rangle}$ *is our usual* K_M.]

5a: *Defn.* Fix a colorset $\mathcal{C} := \{$*blue*, *green*$\}$.

> *A* $\mathcal{C}$***-coloring*** *of* H

is a map $\mathbf{f}:\mathbb{E}\to\mathcal{C}$. Given a subset $V' \subset \mathbb{V}$,

> *let* $\mathbf{f}\lfloor_{V'}$ *mean the coloring* $\mathbf{f}\lfloor_{\mathbb{E}'}$ *defined by* (¥).

Consider a rank $\rho \in [2\,..\,\infty)$ and posint n. Suppose there is a posint M st:

> *For each* $\mathcal{C}$*-coloring of the complete hypergraph* $K_M^{\langle\rho\rangle}$*, there is a cardinality-*n *subset* $W \subset [1\,..\,M]$ *so that coloring* $\mathbf{f}\lfloor_W$ *is constant.*

(IOWords, this edge-colored $K_M^{\langle\rho\rangle}$ admits a monochromatic $K_n^{\langle\rho\rangle}$.) The smallest such M is the ***hypergraph Ramsey number*** $R^{\langle\rho\rangle}(n)$. □

5b: Hypergraph Ramsey Thm. *Fix a rank* $\rho \in \mathbb{Z}_+$. *Then*

i: *Each coloring of* $K_\infty^{\langle\rho\rangle}$ *admits a monochromatic* $K_\infty^{\langle\rho\rangle}$*-subgraph.*

ii: *Each posint* n*: Ramsey number* $R^{\langle\rho\rangle}(n)$ *is finite.*◊

Proof of (ii). The analogous compactness argument of (1c) works here. ♦

Proof of (i). We induct on ρ. We'll show the induction for ρ=5, assuming the ρ=4 case.

Our vertex set is $\mathbb{V} := \{1, 2, \ldots\}$, and we are given a coloring $\mathbf{f}:\mathcal{P}_5(\mathbb{V})\to\mathcal{C}$. Suppose we could find an infinite subset $\mathcal{W} \subset \mathbb{V}$ and a color map $g:\mathcal{P}_4(\mathcal{W})\to\mathcal{C}$ with this property:

†: *For each* $S \in \mathcal{P}_4(\mathcal{W})$ *and each* $y \in \mathcal{W}$ *with* $y > \mathrm{Max}(S)$*, the color* $\mathbf{f}(S \sqcup \{y\})$ *equals* $g(S)$.

The rank=4 case of (5b) asserts there is an ∞-subset $\mathbf{X} \subset \mathcal{W}$ so that

> *Our* g*-coloring is constant on* $\mathcal{P}_4(\mathbf{X})$*; say* blue*.*

Given a $T \in \mathcal{P}_5(\mathbf{X})$, write it as $T = \{w_1, \ldots, w_4, w_5\}$ with $w_1 < \ldots < w_5$. By (†), then,

$$\mathbf{f}(T) = g(\{w_1, \ldots, w_4\}) = \textit{blue}.$$

Hence $\mathbf{f}$ is constant blue on $\mathcal{P}_5(\mathbf{X})$.

Building $\mathcal{W}$. We'll inductively construct vertices $w_1 < w_2 < \ldots$ and infinite $\mathbb{V}$-subsets $\mathbf{Y}_1 \supset \mathbf{Y}_2 \supset \cdots$. Our $\mathcal{W}$ will be $\{w_1, w_2, w_3, \ldots\}$.

Let $w_1 := 1$, $\mathbf{I}_1 := \{w_1\}$ and $\mathbf{Y}_1 := [2\,..\,\infty)$.

At Stage k: We have $\mathbf{I}_k := \{w_1, \ldots, w_k\}$, and a partially-defined $g()$, defined on $\mathcal{P}_4(\mathbf{I}_k)$. We have an infinite vertex-set $\mathbf{Y}_k$, such that:

i: Our $w_k < y$, for each $y \in \mathbf{Y}_k$.

ii: For each $S \in \mathcal{P}_4(\mathbf{I}_k)$, and each $y \in \mathbf{Y}_k$, the color $\mathbf{f}(S \sqcup \{y\})$ equals $g(S)$.

For Stage $[k+1]$, define $w_{k+1} := \mathrm{Min}(\mathbf{Y}_k)$, and temporary set

$$J_0 := \mathbf{Y}_k \smallsetminus \{w_{k+1}\}.$$

Let $S_1, S_2, \ldots, S_L$ be some enumeration of those cardinality-4 subsets of $\mathbf{I}_{k+1}$ that own w_{k+1}.

There is a color, say, blue, and an infinite set of $y \in J_0$, so that $\mathbf{f}(S_1 \sqcup \{y\})$ is blue. Extend $g()$ by defining $g(S_1) :=$ *blue*. Use J_1 for this set of points y.

There is a color, say, green, and an infinite set of $y \in J_1$, so that $\mathbf{f}(S_2 \sqcup \{y\})$ is green. Define $g(S_2) :=$ *green*. Use J_2 for this set of points y.

Continue, until you have shrunk to J_L. Lastly, let $\mathbf{Y}_{k+1} := J_L$. ♦

5c**:** *Defn.* The ***Erdös-Szekeres number*** $\mathrm{ES}(n)$, is the smallest posint M so that *each* collection of M points in the plane with no three colinear, has a subset of n points which form a *convex* n-gon. [Caveat: There are at least two different results called the Erdős-Szekeres thm.] □

5d**:** ES-Theorem. *For the Erdős-Szekeres number,*

$$\mathrm{ES}(n) \leqslant R^{\langle 3 \rangle}(n,n) =: M. \qquad \diamondsuit$$

Proof. With vertex-set $[1 .. M]$, construct a Cyan-Amber-coloring of $K^{\langle 3 \rangle}_{[1..M]}$, as follows: For each triple $\mathbf{u}<\mathbf{v}<\mathbf{w}$ in $[1..M]$, color the $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$-edge Cyan if the $\mathbf{u}\to\mathbf{v}\to\mathbf{w}\to\mathbf{u}$ traversal is ClockWise [CW]; otherwise, paint Amber the $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$-edge, since its traversal is Anti-clockWise [AW].

By hypothesis, there exists an n-set $S \subset [1..M]$, so that all the $K^{\langle 3 \rangle}_S$-edges are, say, CW.

Convex n-gon. FTSOC, suppose the n points of S do *not* form a convex n-gon. Then some point $P \in S$ is in the convex-hull of S. So there are distinct points $\mathbf{u}<\mathbf{v}<\mathbf{w}$ in S, with an S-point $P \in \mathrm{Hull}(\{\mathbf{u},\mathbf{v},\mathbf{w}\})$.

Recall $\mathbf{u}\to\mathbf{v}\to\mathbf{w}$ is CW. We must have $P > \mathbf{u}$; else $P\to\mathbf{u}\to\mathbf{w}$ is AW.

And $P \overset{\text{must}}{>} \mathbf{v}$; else $\mathbf{u}\to P\to\mathbf{v}$ is AW. Continuing, $P \overset{\text{must}}{>} \mathbf{w}$; else $\mathbf{v}\to P\to\mathbf{w}$ is AW. But now, $\mathbf{u}\to\mathbf{w}\to P$ is AW. ※ ♦

§A Hales-Jewett

The Statement

Suppose we have a finite alphabet $\mathbb{A} = \{\mathtt{a}, \mathtt{b}, \dots\}$ and we fix a length $\mathbf{h}$. A ***degree-$\mathfrak{D}$ polynomial*** $f(x_1, \dots, x_{\mathfrak{D}})$ over $\mathbb{A}^{\times \mathbf{h}}$ is a word

6: $$f \in \left[\mathbb{A} \sqcup \{x_1, \dots, x_{\mathfrak{D}}\}\right]^{\times \mathbf{h}}$$

where each variable x_j occurs at least once in f. We evaluate $f()$ at a $\mathfrak{D}$-tuple of $\mathbb{A}$-letters by plugging them in for the $\mathfrak{D}$ variables. The range of this polynomial is a subset of $\mathbb{A}^{\times \mathbf{h}}$ and has $|\mathbb{A}|^{\mathfrak{D}}$ members. This range is called a ***$\mathfrak{D}$-dimensional (affine) subspace***. The RHS of (6) implies

7: *There are at most* $\left[|\mathbb{A}| + \mathfrak{D}\right]^{\mathbf{h}}$ *many $\mathfrak{D}$-dimensional subspaces of* $\mathbb{A}^{\times \mathbf{h}}$.

By the way, a 1-dimensional subspace is also called an (affine) ***line***.

The Hales-Jewett theorem states that given an ***alphabet size*** $\alpha := |\mathbb{A}|$, a number μ of colors and a dimension $\mathfrak{D}$:

> *There is a function* $\mathbf{h} = \mathbf{h}(\mathfrak{D}, \mu, \alpha)$ *so that each μ-coloring (coloring by μ many colors) of the set of words* $\mathbb{A}^{\times \mathbf{h}}$ *will have a monochromatic $\mathfrak{D}$-dimensional subspace.*

One cannot guarantee the stronger statement that there is a monochromatic subspace parallel to the coordinate axes ie. where each variable in the word of (6) occurs exactly once. This is already false in the $\mathfrak{D}$=1 case: Let the ***color*** of a word in $\{\mathtt{0}, \mathtt{1}\}^{\times \mathbf{h}}$ be the mod-2 sum of its bits. Then a ***line*** consists of a pair of $\mathbf{h}$-words $u\mathtt{0}w$ and $u\mathtt{1}w$ differing in a single bit-position —which therefore have different colors.

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As of: *Tuesday 17Apr2018.* Typeset: *24Apr2018* at *10:21.*