

Ramsey-like theorems

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ABSTRACT: Ramsey-like theorems. Hales-Jewett.

In this note, a **coloring** is an edge-coloring unless specified otherwise.

Entrance. (jk: Jeudi, 26 mars, 1985) Let K_n denote the complete graph on n vertices. Let $\mathcal{R}(n)$ denote the n^{th} **Ramsey number**, that is, the smallest M such that if one colors each edge of K_M either **blue** or **green**, then there must exist a monochromatic subgraph isomorphic to K_n .

Notation. For vertex \mathbf{w} in a blue-green graph, let $bV(\mathbf{w})$ be the set of vertices connected to \mathbf{w} by a **blue** edge; use $gV(\mathbf{w})$ for those connected to \mathbf{w} by **green** edges. We thus have vertex degrees

$$b\text{Deg}(\mathbf{w}) := |bV(\mathbf{w})| \quad \text{and} \quad g\text{Deg}(\mathbf{w}) := |gV(\mathbf{w})|.$$

Hence $b\text{Deg}(\mathbf{w}) + g\text{Deg}(\mathbf{w}) = \text{Deg}(\mathbf{w})$.

1a: Theorem. For $n = 1, 2, 3, \dots$,

$$1a': \quad \mathcal{R}(n+1) \leq 2[1 + [n-1]\mathcal{R}(n)]. \quad \diamond$$

Proof. With $M := \text{RhS}(1a')$, let \mathbb{V} denote the vertex set of K_M . FTSOC, assume we have edge-colored K_M so that it has *no* monochromatic copy of K_{n+1} .

Pick a $\mathbf{u}_0 \in \mathbb{V}$. WLOGenerality at least half of the edges from \mathbf{u}_0 are **blue**. Letting $B_0 := b\text{Deg}(\mathbf{u}_0)$, we have

$$*: \quad B_0 \geq \left\lceil \frac{M-1}{2} \right\rceil \stackrel{\text{note}}{=} 1 + [n-1] \cdot \mathcal{R}(n).$$

Building a blue complete-subgraph. Pick a vertex $\mathbf{u}_1 \in bV(\mathbf{u}_0)$. The set $bV(\mathbf{u}_0) \setminus \{\mathbf{u}_1\}$ can have at most $\mathcal{R}(n) - 1$ vertices not belonging to $bV(\mathbf{u}_1)$. If otherwise, then there would be a copy of $K_{\mathcal{R}(n)}$ in our graph, disjoint from \mathbf{u}_0 and \mathbf{u}_1 , such that each vertex in this copy was connected to \mathbf{u}_0 by a **blue** edge and to \mathbf{u}_1 by a **green** edge. This copy would, by hypothesis, contain a monochromatic copy –call it H – of K_n . Were H **blue**, then $H \sqcup \{\mathbf{u}_0\}$ is a **blue** K_{n+1} . Else, H is **green**, so $H \sqcup \{\mathbf{u}_1\}$ is a **green** K_{n+1} . Either is a \otimes .

Consequently the set $bV(\mathbf{u}_0) \cap bV(\mathbf{u}_1)$ has at least

$$|bV(\mathbf{u}_0)| - |\{\mathbf{u}_1\}| - [\mathcal{R}(n) - 1] \stackrel{\text{note}}{=} B_0 - \mathcal{R}(n)$$

many vertices. Pick some \mathbf{u}_2 in $bV(\mathbf{u}_0) \cap bV(\mathbf{u}_1)$. The same argument shows $bV(\mathbf{u}_0) \cap bV(\mathbf{u}_1) \cap bV(\mathbf{u}_2)$ has at least

$$[B_0 - \mathcal{R}(n)] - \mathcal{R}(n)$$

many vertices.

Continue. At stage $n-1$ we will have chosen distinct vertices $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{n-1}$. Further, the intersection of their **blue** vertex-sets will satisfy

$$|bV(\mathbf{u}_0) \cap bV(\mathbf{u}_1) \cap \dots \cap bV(\mathbf{u}_{n-1})| \geq B_0 - [n-1]\mathcal{R}(n)$$

and hence be non-empty, by (*). Picking a vertex \mathbf{u}_n in this intersection, now every pair of vertices in $\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n\}$ is connected by a **blue** edge. $\otimes \blacklozenge$

Remark. Values $\mathcal{R}(1)$, $\mathcal{R}(2)$ and $\mathcal{R}(3)$ are 1, 2 and 6 respectively. Using the above theorem, easily

$$\mathcal{R}(n+1) \leq [n! \cdot 2^n]. \quad \square$$

More colors. If we allow $\mu \in \mathbb{Z}_+$ many colors, then we get a corresponding sequence of Ramsey numbers $\mathcal{R}_\mu(n)$ for $n = 1, 2, \dots$. Here, $\mathcal{R}_1(n) = n$, and $\mathcal{R}_2(\cdot)$ is another name for $\mathcal{R}(\cdot)$.

For each $\mu > 1$, lump the μ colors into two **super-colors** consisting each of at most $h := \lceil \mu/2 \rceil$ colors. Applying $\mathcal{R}(\cdot)$ to these two supercolors, implies that

$$\mathcal{R}_\mu(n) \leq \mathcal{R}(\mathcal{R}_h(n)) \leq \mathcal{R}(\mathcal{R}(\dots \mathcal{R}(n) \dots)),$$

where the RhS has $\lceil \log_2(\mu) \rceil$ occurrences of $\mathcal{R}(\cdot)$. \square

Infinite Ramsey's Theorem

(jk, Feb1989) Let K_∞ denote the complete graph on denumerably many vertices.

1b: Infinite Ramsey's Theorem. Each edge \overline{uv} of $H = (\mathbb{V}, \mathbb{E})$, a K_∞ graph, is colored either blue or green. Then H includes a monochromatic K_∞ . \diamond

Proof. Let $W_0 := \mathbb{V}$. At stage N we have vertex-sets

$$W_0 \supset W_1 \supset \dots \supset W_j \supset \dots \supset W_N,$$

each infinite. Moreover, we have vertices $(\mathbf{u}_j)_{j=0}^{N-1}$, with $\mathbf{u}_j \in W_j \setminus W_{j+1}$, so that each edge-set

$$\mathbb{E}_j := \{ \overline{v\mathbf{u}_j} \mid v \in W_{j+1} \}$$

is monochromatic.

Continue the induction by picking an arbitrary vertex $\mathbf{u}_N \in W_N$. Since $W_N \setminus \{ \mathbf{u}_N \}$ is infinite, it includes an infinite set of vertices \mathbf{v} so that $\{ \overline{v\mathbf{u}_N} \}_v$ is monochromatic. Define W_{N+1} to be this infinite set of vertices.

An infinite monochromatic sequence. Each of the sets $\mathbb{E}_0, \mathbb{E}_1, \dots$ is either blue or green. Thus there is a subsequence $N_1 < N_2 < \dots$ so that, WLOG, each \mathbb{E}_{N_j} is green. The consequence is that the complete graph with vertex-set $(\mathbf{u}_{N_j})_{j=1}^\infty$ has every edge green. \blacklozenge

1c: Observation. Infinite Ramsey's Thm implies the Finite Ramsey's Thm, by a compactness argument. \diamond

Proof. Consider K_∞ , the complete graph with vertex-set $\mathbb{V} = \{1, 2, \dots\}$.

FTSOC suppose the, say, 5th Ramsey number were not finite. Then for each n , there would exist a blue-green coloring of the edges on vertex-set $\{1, \dots, n\}$ –call this colored graph H_n – so that H_n does not include a monochromatic copy of K_5 .

Interpret H_n as an edge-coloring of K_∞ : Color plum each edge that is not between vertices $[1..n]$. (I.e, for posints $\mathbf{u} < \mathbf{v}$: If $\mathbf{v} > n$ then color edge $\overline{u\mathbf{v}}$ plum.)

We now have a sequence H_1, H_2, H_3, \dots of $\{blue, green, plum\}$ colorings of K_∞ . Since each edge

of K_∞ has only finitely many [three] possible colors, the sequence of colorings has a convergent subsequence $(H_{n_j})_{j=1}^\infty$. The coloring obtained in the limit, an edge-coloring of K_∞ , has *no plum*. But ∞ -Ramsey's-Thm guarantees a monochromatic copy of K_∞ ; WLOG it is green. Letting $\mathbf{u}_1 < \mathbf{u}_2 < \dots < \mathbf{u}_5$ be its first 5 vertices, this green K_5 was *already* a subgraph of $H_{\mathbf{u}_5}$. \otimes

2a: Two-variable $\mathcal{R}()$. For posints \mathbf{b}, \mathbf{g} , let $\mathcal{R}(\mathbf{b}, \mathbf{g})$ be the smallest M st. each blue-green-coloring of K_M has either a blue K_b or a green K_g . So $\mathcal{R}(n) = \mathcal{R}(n, n)$.

Evidently, $\mathcal{R}(\mathbf{b}, 1) = 1$ and $\mathcal{R}(\mathbf{b}, 2) = \mathbf{b}$; and this holds symmetrically, since $\mathcal{R}(\mathbf{b}, \mathbf{g}) = \mathcal{R}(\mathbf{g}, \mathbf{b})$. \square

2b: Theorem. For all $\mathbf{b}, \mathbf{g} \in \mathbb{Z}_+$,

$$*: \quad \mathcal{R}(\mathbf{b}, \mathbf{g}) \leq \mathcal{R}(\mathbf{b}-1, \mathbf{g}) + \mathcal{R}(\mathbf{b}, \mathbf{g}-1). \quad \diamond$$

Proof. Let $M := \text{RhS}(*).$ Fix a vertex \mathbf{w} ; it has $M-1$ edges, so has either at least $\mathcal{R}(\mathbf{b}-1, \mathbf{g})$ blue-edges, or at least $\mathcal{R}(\mathbf{b}, \mathbf{g}-1)$ green-edges. WLOG

$$|bV(\mathbf{w})| \geq \mathcal{R}(\mathbf{b}-1, \mathbf{g}). \quad \left[\text{Recall } bV(\mathbf{w}) \text{ is the set of vertices blue-connected to } \mathbf{w}. \right]$$

If the graph induced by $bV(\mathbf{w})$ has a green K_g , then DONE. Else, the induced graph admits a blue $K_{\mathbf{b}-1}$ which, together with vertex \mathbf{w} , induces a blue K_b . \blacklozenge

2c: Exercise: Coloring with μ colors, let $\mathcal{R}(n_1, \dots, n_\mu)$ be the smallest M st. each μ -coloring, H , of K_M has an index j for which H admits a K_{n_j} with all edges the j^{th} -color. **Exer-E1:** What is the μ -color analog of Thm 2b? [Hint: Don't jump to conclusions.] \square

Examples from Bona's text

In class we proved $\mathcal{R}(3)=6$. In this instance (2b*) is sharp, as

$$\mathcal{R}(3,3) \leq 2 \cdot \mathcal{R}(3,2) = 2 \cdot 3 = 6.$$

Using Thm 2b again,

$$\mathcal{R}(3,4) \leq \mathcal{R}(2,4) + \mathcal{R}(3,3) = 4 + 6 = 10.$$

3a: Claim: $\mathcal{R}(3,4) = 9$. ♦

Pf of $\mathcal{R}(3,4) \leq 9$. FTSOC, suppose H is a blue-green-coloring of K_9 with no blue triangle, nor green K_4 .

Could *some* vertex \mathbf{w} have $\text{bDeg}(\mathbf{w}) \geq 4$? Since no blue triangle, no pair of vertices in $bV(\mathbf{w})$ has a blue-edge; but then $bV(\mathbf{w})$ induces a green K_4 . ✘

Could *every* \mathbf{w} have $\text{bDeg}(\mathbf{w}) = 3$? Then the blue-degree-sum is $3 \cdot 9$, which is odd. But this degree-sum must also equal twice the number of blue edges. ✘

So *there exists* a vertex \mathbf{w} with $\text{bDeg}(\mathbf{w}) \leq 2$, hence

$$|gV(\mathbf{w})| \geq 8 - 2 = 6.$$

Since $\mathcal{R}(3) = 6$, and $gV(\mathbf{w})$ cannot have a blue triangle, it must have a green triangle, G . But then $G \sqcup \{\mathbf{w}\}$ is a green K_4 . ♦

Pf of $\mathcal{R}(3,4) > 8$. On vertex-set $\mathbb{V} := [0..8)$, let \equiv and \oplus and \ominus each operate mod 8.

Connect vertices $j, k \in \mathbb{V}$ by blue IFF $k \ominus j$ is either

- 3, producing a blue octagon, **or**
- 4, producing four center-crossing blue edges. ♦

But no triple of numbers from $\{3, 4\}$ has \oplus -sum equal to zero. Hence there is *no* blue triangle.

Color green the remaining edges, i.e $k \ominus j \in \{1, 2\}$. Difference=2 makes green square $\{0, 2, 4, 6\}$, and square $\{1, 3, 5, 7\}$. And difference=1 make a green octagon $\{0, 1, 2, 3, 4, 5, 6, 7\}$.

FTSOCcontradiction, suppose G is a green K_4 subgraph. It can't have three vertices in one square [diff=4], so G has *two* vertices in *each* square, say $\{0, 2\}$ and $\{\mathbf{v}, \mathbf{w}\}$. But \mathbf{v} must differ from each of 0,2 by 1 or 2; so $\mathbf{v} \stackrel{\text{must}}{=} 1$. Ditto, $\mathbf{w} = 1$. ✘

The result now allows, courtesy Thm 2b, that

$$\mathcal{R}(4,4) \leq 2 \cdot \mathcal{R}(3,4) = 18.$$

3b: Claim: $\mathcal{R}(4,4) = 18$. ♦

Proof. To show $\mathcal{R}(4,4) > 17$, we exhibit a blue-green-coloring of K_{17} with no monochromatic K_4 .

On vertex-set $\mathbb{V} := \{0, 1, \dots, 8, -8, -7, \dots, -1\}$, let \equiv , \oplus , \ominus and $\langle \cdot \rangle$ each operate mod 17. Blue-connect $j, k \in \mathbb{V}$ IFF $k \ominus j$ is a mod-17 QR [Quadratic Residue].

	x	$\langle x^2 \rangle$	x	$\langle x^2 \rangle$
3c:	± 1	1	± 5	8
	± 2	4	± 6	2
	± 3	-8	± 7	-2
	± 4	-1	± 8	-4

Thus QR = $\{\pm 1, \pm 2, \pm 4, \pm 8\}$; so our edge-lengths are $\{1, 2, 4, 8\}$. And NQR = $\{\pm 3, \pm 5, \pm 6, \pm 7\}$, giving rise to the green edges. Multiplying the vertices by a non-QR element, will exchange the blue and green edges [using that mult distributes-over addition]. So: *It suffices to show that there is no blue K_4 .*

Number Thy. FTSOC, suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{V}$ are the vertices of a blue K_4 . WLOG $\mathbf{a}=0$; replace vertex x by $x \ominus \mathbf{a}$. WLOG $\mathbf{b}=1$; replace each x by $\langle x/\mathbf{b} \rangle$. [The mapping $x \mapsto \langle x/\mathbf{b} \rangle$ is color-preserving, since \mathbf{b} is a QR.]

The vertices are now $\{0, 1, \mathbf{c}, \mathbf{d}\}$.

Vertex \mathbf{c} must differ from 0 and 1 by values in $\{1, 2, 4, 8\}$; so \mathbf{c} is 2 or -1; WLOG $\mathbf{c}=2$. Vertex \mathbf{d} , relative to vertices 1 and 2, must be 3 or 0; but 0 is taken. Our supposedly-blue K_4 is $\{0, 1, 2, 3\}$.

Alas, edge $0\bar{3}$ is green. ✘ ♦

van der Waerden's thm

A *shift* $\mathbf{s} \in \mathbb{Z}$ and a *gap* $\mathbf{g} \in \mathbb{Z}_+$, give rise to *finite* and *infinite arithmetic progressions*:

$$\begin{aligned} \mathbb{A}_n(\mathbf{s}, \mathbf{g}) &:= \{\mathbf{s} + j\mathbf{g} \mid j \in [0..n)\}; \\ \mathbb{A}(\mathbf{s}, \mathbf{g}) = \mathbb{A}_\infty(\mathbf{s}, \mathbf{g}) &:= \{\mathbf{s} + j\mathbf{g} \mid j \in \mathbb{Z}\}. \end{aligned}$$

Use \mathbb{A}_n to refer to *some* n -term A.P, when the shift and gap are irrelevant.

4a: van der Waerden theorem. *For each posint n , there is a finite M so that each 2-coloring (of the elements) of \mathbb{A}_M , produces a monochromatic \mathbb{A}_n .* \diamond

The smallest such M is written $\mathcal{W}(n)$. For example, $\mathcal{W}(3) = 9$. If μ many colors are allowed, then we write $\mathcal{W}_\mu(n)$ for the minimum M .

The infinite analog of vdW thm fails, as shown next.

4b: Obs. *There exists a 2-coloring of \mathbb{Z} which admits no monochromatic sub- \mathbb{A}_∞ .* \diamond

Proof. We produce a coloring for which the only monochromatic arithmetic-progs are finite.

Take a surjection $n \mapsto (\mathbf{s}_n, \mathbf{g}_n)$ from $\mathbb{Z}_+ \rightarrow \mathbb{Z} \times \mathbb{Z}_+$, and let $\mathcal{A}(n) := \mathbb{A}_\infty(\mathbf{s}_n, \mathbf{g}_n)$.

We color \mathbb{Z} in stages. At stage n : Paint one **blue** and one **green**, the two posints in $\mathcal{A}(n)$ which are closest to zero, and were not yet colored. Then paint one **blue** and one **green**, the two negints in $\mathcal{A}(n)$ which are closest to zero, and were not yet colored.

Send $n \nearrow \infty$. Finally, color arbitrarily the remaining uncolored integers. \blacklozenge

Hypergraphs

Let $\mathcal{P}_4(\mathbb{V})$ be the collection of cardinality-4 subsets of \mathbb{V} ; so it has $\binom{|\mathbb{V}|}{4}$ many members. A “4-**hypergraph** on \mathbb{V} ” is a pair $H := (\mathbb{V}, \mathbb{E})$ where $\mathbb{E} \subset \mathcal{P}_4(\mathbb{V})$ is the set of **hyper-edges** of H . This cardinality, 4, is called the **rank** of H .

A subset $V' \subset \mathbb{V}$ determines a sub-hypergraph $H' := (V', \mathbb{E}')$ of H , where

$$\mathbb{E}': \quad \mathbb{E}' := \{S \in \mathbb{E} \mid S \subset V'\}.$$

By the way, the “*complete* 4-hypergraph on \mathbb{V} ” is $(\mathbb{V}, \mathcal{P}_4(\mathbb{V}))$. Setting $M := |\mathbb{V}|$, we’ll use $K_M^{(4)}$ to refer to this hypergraph. [So $K_M^{(2)}$ is our usual K_M .]

5a: *Defn.* Fix a colorset $\mathcal{C} := \{\text{blue}, \text{green}\}$.

A \mathcal{C} -coloring of H

is a map $\mathbf{f}: \mathbb{E} \rightarrow \mathcal{C}$. Given a subset $V' \subset \mathbb{V}$,

let $\mathbf{f}|_{V'}$ mean the coloring $\mathbf{f}|_{\mathbb{E}'}$ defined by (5).

Consider a rank $\rho \in [2.. \infty)$ and posint n . Suppose there is a posint M st:

For each \mathcal{C} -coloring of the complete hypergraph $K_M^{(\rho)}$, there is a cardinality- n subset $W \subset [1.. M]$ so that coloring $\mathbf{f}|_W$ is constant.

(IOWords, this edge-colored $K_M^{(\rho)}$ admits a monochromatic $K_n^{(\rho)}$.) The smallest such M is the **hypergraph Ramsey number** $R^{(\rho)}(n)$. \square

5b: **Hypergraph Ramsey Thm.** Fix a rank $\rho \in \mathbb{Z}_+$. Then

- i: Each coloring of $K_\infty^{(\rho)}$ admits a monochromatic $K_\infty^{(\rho)}$ -subgraph.
- ii: Each posint n : Ramsey number $R^{(\rho)}(n)$ is finite. \diamond

Proof of (ii). The analogous compactness argument of (1c) works here. \diamond

Proof of (i). We induct on ρ . We’ll show the induction for $\rho=5$, assuming the $\rho=4$ case.

Our vertex set is $\mathbb{V} := \{1, 2, \dots\}$, and we are given a coloring $\mathbf{f}: \mathcal{P}_5(\mathbb{V}) \rightarrow \mathcal{C}$. Suppose we could find an infinite subset $\mathcal{W} \subset \mathbb{V}$ and a color map $g: \mathcal{P}_4(\mathcal{W}) \rightarrow \mathcal{C}$ with this property:

†: For each $S \in \mathcal{P}_4(\mathcal{W})$ and each $y \in \mathcal{W}$ with $y > \text{Max}(S)$, the color $\mathbf{f}(S \sqcup \{y\})$ equals $g(S)$.

The rank=4 case of (5b) asserts there is an ∞ -subset $\mathbf{X} \subset \mathcal{W}$ so that

Our g -coloring is constant on $\mathcal{P}_4(\mathbf{X})$; say *blue*.

Given a $T \in \mathcal{P}_5(\mathbf{X})$, write it as $T = \{w_1, \dots, w_4, w_5\}$ with $w_1 < \dots < w_5$. By (†), then,

$$\mathbf{f}(T) = g(\{w_1, \dots, w_4\}) = \text{blue}.$$

Hence \mathbf{f} is constant *blue* on $\mathcal{P}_5(\mathbf{X})$.

Building \mathcal{W} . We’ll inductively construct vertices $w_1 < w_2 < \dots$ and infinite \mathbb{V} -subsets $\mathbf{Y}_1 \supset \mathbf{Y}_2 \supset \dots$. Our \mathcal{W} will be $\{w_1, w_2, w_3, \dots\}$.

Let $w_1 := 1$, $\mathbf{I}_1 := \{w_1\}$ and $\mathbf{Y}_1 := [2.. \infty)$.

At **STAGE k** : We have $\mathbf{I}_k := \{w_1, \dots, w_k\}$, and a partially-defined $g()$, defined on $\mathcal{P}_4(\mathbf{I}_k)$. We have an infinite vertex-set \mathbf{Y}_k , such that:

- i: Our $w_k < y$, for each $y \in \mathbf{Y}_k$.
- ii: For each $S \in \mathcal{P}_4(\mathbf{I}_k)$, and each $y \in \mathbf{Y}_k$, the color $\mathbf{f}(S \sqcup \{y\})$ equals $g(S)$.

For **STAGE $[k+1]$** , define $w_{k+1} := \text{Min}(\mathbf{Y}_k)$, and temporary set

$$J_0 := \mathbf{Y}_k \setminus \{w_{k+1}\}.$$

Let S_1, S_2, \dots, S_L be some enumeration of those cardinality-4 subsets of \mathbf{I}_{k+1} that own w_{k+1} .

There is a color, say, *blue*, and an infinite set of $y \in J_0$, so that $\mathbf{f}(S_1 \sqcup \{y\})$ is *blue*. Extend $g()$ by defining $g(S_1) := \text{blue}$. Use J_1 for this set of points y .

There is a color, say, *green*, and an infinite set of $y \in J_1$, so that $\mathbf{f}(S_2 \sqcup \{y\})$ is *green*. Define $g(S_2) := \text{green}$. Use J_2 for this set of points y .

Continue, until you have shrunk to J_L . Lastly, let $\mathbf{Y}_{k+1} := J_L$. \diamond

5c: Defn. The **Erdős-Szekeres number** $ES(n)$, is the smallest posint M so that *each* collection of M points in the plane with no three colinear, has a subset of n points which form a *convex* n -gon. [Caveat: There are at least two different results called the Erdős-Szekeres thm.] \square

5d: ES-Theorem. For the Erdős-Szekeres number,

$$ES(n) \leq R^{(3)}(n, n) =: M. \quad \diamond$$

Proof. With vertex-set $[1..M]$, construct a **Cyan-Amber**-coloring of $K_{[1..M]}^{(3)}$, as follows: For each triple $\mathbf{u} < \mathbf{v} < \mathbf{w}$ in $[1..M]$, color the $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ -edge **Cyan** if the $\mathbf{u} \rightarrow \mathbf{v} \rightarrow \mathbf{w} \rightarrow \mathbf{u}$ traversal is ClockWise [**CW**]; otherwise, paint **Amber** the $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ -edge, since its traversal is Anti-clockWise [**AW**].

By hypothesis, there exists an n -set $S \subset [1..M]$, so that all the $K_S^{(3)}$ -edges are, say, **CW**.

Convex n -gon. FTSOC, suppose the n points of S do *not* form a convex n -gon. Then some point $P \in S$ is in the convex-hull of S . So there are distinct points $\mathbf{u} < \mathbf{v} < \mathbf{w}$ in S , with an S -point $P \in \text{Hull}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$.

Recall $\mathbf{u} \rightarrow \mathbf{v} \rightarrow \mathbf{w}$ is **CW**. We must have $P > \mathbf{u}$; else $P \rightarrow \mathbf{u} \rightarrow \mathbf{w}$ is **AW**.

And $P \overset{\text{must}}{>} \mathbf{v}$; else $\mathbf{u} \rightarrow P \rightarrow \mathbf{v}$ is **AW**. Continuing, $P \overset{\text{must}}{>} \mathbf{w}$; else $\mathbf{v} \rightarrow P \rightarrow \mathbf{w}$ is **AW**. But now, $\mathbf{u} \rightarrow \mathbf{w} \rightarrow P$ is **AW**. \otimes \blacklozenge

§A Hales-Jewett

The Statement

Suppose we have a finite alphabet $\mathbb{A} = \{a, b, \dots\}$ and we fix a length \mathbf{h} . A *degree- \mathfrak{D} polynomial* $f(x_1, \dots, x_{\mathfrak{D}})$ over $\mathbb{A}^{\times \mathbf{h}}$ is a word

$$6: \quad f \in [\mathbb{A} \sqcup \{x_1, \dots, x_{\mathfrak{D}}\}]^{\times \mathbf{h}}$$

where each variable x_j occurs at least once in f . We evaluate $f()$ at a \mathfrak{D} -tuple of \mathbb{A} -letters by plugging them in for the \mathfrak{D} variables. The range of this polynomial is a subset of $\mathbb{A}^{\times \mathbf{h}}$ and has $|\mathbb{A}|^{\mathfrak{D}}$ members. This range is called a *\mathfrak{D} -dimensional (affine) subspace*. The RHS of (6) implies

$$7: \quad \text{There are at most } [|\mathbb{A}| + \mathfrak{D}]^{\mathbf{h}} \text{ many } \mathfrak{D}\text{-dimensional subspaces of } \mathbb{A}^{\times \mathbf{h}}.$$

By the way, a 1-dimensional subspace is also called an (affine) *line*.

The Hales-Jewett theorem states that given an *alphabet size* $\alpha := |\mathbb{A}|$, a number μ of colors and a dimension \mathfrak{D} :

There is a function $\mathbf{h} = \mathbf{h}(\mathfrak{D}, \mu, \alpha)$ so that each μ -coloring (coloring by μ many colors) of the set of words $\mathbb{A}^{\times \mathbf{h}}$ will have a monochromatic \mathfrak{D} -dimensional subspace.

One cannot guarantee the stronger statement that there is a monochromatic subspace parallel to the coordinate axes ie. where each variable in the word of (6) occurs exactly once. This is already false in the $\mathfrak{D}=1$ case: Let the *color* of a word in $\{0, 1\}^{\times \mathbf{h}}$ be the mod-2 sum of its bits. Then a *line* consists of a pair of \mathbf{h} -words $u0w$ and $u1w$ differing in a single bit-position—which therefore have different colors.