

Number Sets. Expression $k \in \mathbb{N}$ [read as “ k is an element of \mathbb{N} ” or “ k in \mathbb{N} ”] means that k is a natural number; a *natnum*. Expression $\mathbb{N} \ni k$ [read as “ \mathbb{N} owns k ”] is a synonym for $k \in \mathbb{N}$.

\mathbb{N} = natural numbers = $\{0, 1, 2, \dots\}$.

\mathbb{Z} = integers = $\{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the *posints*, use \mathbb{Z}_+ . Use \mathbb{Z}_- for the negative integers, the *negints*.

\mathbb{Q} = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use \mathbb{Q}_+ for the positive rationals and \mathbb{Q}_- for the negative rationals.

\mathbb{R} = reals. The *posreals* \mathbb{R}_+ and the *negreals* \mathbb{R}_- .

\mathbb{C} = complex numbers, also called the *complexes*.

For $\omega \in \mathbb{C}$, let “ $\omega > 5$ ” mean “ ω is real and $\omega > 5$ ”.

[Use the same convention for $\geq, <, \leq$, and also if 5 is replaced by any real number.]

An “*interval of integers*” $[b..c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[e..2\pi] = \{3, 4, 5, 6\} = [3..6] = (2..6]$. We allow b and c to be $\pm \infty$; so $(-\infty..-1]$ is \mathbb{Z}_- . And $[-\infty..-1]$, is $\{-\infty\} \cup \mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3, \lfloor -\pi \rfloor = -4$.
Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$
and $|-5 + 2i| = \sqrt{29}$.

Mathematical objects. Seq: ‘sequence’. poly(s): ‘polynomial(s)’. irred: ‘irreducible’. Coeff: ‘coefficient’ and var(s): ‘variable(s)’ and parm(s): ‘parameter(s)’. Expr.: ‘expression’. Fnc: ‘function’ (so ratfnc: means rational function, a ratio of polynomials). cty: ‘continuity’. cts: ‘continuous’. diff’able: ‘differentiable’. CoV: ‘Change-of-Variable’. Col: ‘Constant of Integration’. Lol: ‘Limit(s) of Integration’. RoC: ‘Radius of Convergence’.

Soln: ‘Solution’. Thm: ‘Theorem’. Prop’n: ‘Proposition’. CEX: ‘Counterexample’. eqn: ‘equation’. RhS: ‘RightHand Side’ of an eqn or inequality. LhS: ‘left-hand side’. Sqrt or Sroot: ‘square-root’, e.g, “the sqroot of 16 is 4”. Ptn: ‘partition’, but pt: ‘point’, as in “a fixed-pt of a map”.

FTC: ‘Fund. Thm of Calculus’. IVT: ‘intermediate-Value Thm’. MVT: ‘Mean-Value Thm’.

The *logarithm* fnc, defined for $x > 0$, is $\log(x) := \int_1^x \frac{dv}{v}$. Its inverse-fnc is $\exp()$. For

$x > 0$, then, $\exp(\log(x)) = x = e^{\log(x)}$. For real t , naturally, $\log(\exp(t)) = t = \log(e^t)$.

PolyExp: ‘Polynomial-times-exponential’; e.g, $[3 + t^2] \cdot e^{4t}$. PolyExp-sum: ‘Sum of polyexps’. E.g, $f(t) := 3te^{2t} + [t^2] \cdot e^t$ is a polyexp-sum.

Phrases. WLOG: ‘Without loss of generality’. IFF: ‘if and only if’. TFAE: ‘The following are equivalent’. ITOF: ‘In Terms Of’. OTForm: ‘of the form’. FTSOC: ‘For the sake of contradiction’. And \otimes = “Contradiction”.

IST: ‘It Suffices to’ as in ISTShow, ISTExhibit.

Use w.r.t: ‘with respect to’ and s.t: ‘such that’.

Latin: e.g: *exempli gratia*, ‘for example’. i.e: *id est*, ‘that is’. N.B: *Nota bene*, ‘Note well’. inter alia: ‘among other things’. QED: *quod erat demonstrandum*, meaning “end of proof”.

Bi/Multi-nomial coeffs. For a natnum n , use “ $n!$ ” to mean “ n factorial”; the product of all posints $\leq n$. So $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 120$. Also $0! = 1 = 1!$.

For natnum B and arb. complex number α , define

Rising Fctrl: $[\alpha \uparrow B] := \alpha \cdot [\alpha + 1] \cdot [\alpha + 2] \cdots [\alpha + [B-1]]$,
Falling Fctrl: $[\alpha \downarrow B] := \alpha \cdot [\alpha - 1] \cdot [\alpha - 2] \cdots [\alpha - [B-1]]$.

E.g, $[[B \downarrow B] = B! = [1 \uparrow B]]$. Two further examples,

$$\left[\left[\frac{2}{7} \downarrow 4 \right] \right] = \frac{2}{7} \cdot \frac{-5}{7} \cdot \frac{-12}{7} \cdot \frac{-19}{7} \text{ and } [[1 \downarrow 3]] = 1 \cdot 0 \cdot -1 = 0.$$

In particular, for $n \in \mathbb{N}$: If $B > n$ then $[[n \downarrow B]] = 0$.

We pronounce $[[5 \downarrow B]]$ as “5 falling-factorial B”.

Binomial. The *binomial coefficient* $\binom{7}{3}$, read “7 choose 3”, means the number of ways of choosing 3 objects from 7 distinguishable objects. Emphasising putting 3 objects in our left pocket and the remaining 4 in our right, we may write the coeff as $\binom{7}{3,4}$. [Read as “7 choose 3-comma-4.”] Evidently

$$\dagger: \binom{N}{j} \xrightarrow{\text{with } k := N-j} \binom{N}{j, k} = \frac{N!}{j! \cdot k!} = \frac{[[N \downarrow j]]}{j!}.$$

Note $\binom{7}{0} = \binom{7}{7} = 1$. Finally, the Binomial theorem says

$$\pounds: [x + y]^N = \sum_{j+k=N} \binom{N}{j, k} \cdot x^j y^k,$$

where (j, k) ranges over all ordered pairs of natural numbers with sum N .

For natnum N , binomials satisfy this addition law:

$$*: \binom{N+1}{B+1} = \overbrace{\binom{N}{B}}^{\text{Pick last object.}} + \overbrace{\binom{N}{B+1}}^{\text{Avoid last object.}}.$$

Extending this to all $B \in \mathbb{Z}$ forces:

$$\binom{N}{B} = 0, \quad \text{when } B > N \text{ or } B \text{ negative.}$$

Case $B > N$ is automatic in formula $\binom{N}{B} = \frac{[[N \downarrow B]]}{B!}$.

Multinomial. In general, for natnums $N = k_1 + \dots + k_P$, the *multinomial coefficient* $\binom{N}{k_1, k_2, \dots, k_P}$ is the number of ways of partitioning N objects, by putting k_1 objects in pocket-one, k_2 objects in pocket-two, ... putting k_P objects in the P^{th} pocket. Easily

$$\dagger: \binom{N}{k_1, k_2, \dots, k_P} = \frac{N!}{k_1! \cdot k_2! \cdot \dots \cdot k_P!}.$$

And $[x_1 + \dots + x_P]^N$ indeed equals the sum of terms

$$\pounds\pounds: \binom{N}{k_1, \dots, k_P} \cdot x_1^{k_1} \cdot x_2^{k_2} \cdots x_P^{k_P},$$

taken over all natnum-tuples $\vec{k} = (k_1, \dots, k_P)$ that sum to N .

Define the sum $S_\ell := k_1 + k_2 + \dots + k_\ell$. Then multinomial LhS(\dagger) equals this product of binomials:

$$\binom{N}{k_1} \cdot \binom{N - S_1}{k_2} \cdot \binom{N - S_2}{k_3} \cdots \binom{N - S_{P-1}}{k_P}.$$

[The last term is $\binom{k_P}{k_P} \stackrel{\text{note}}{=} 1$.]

Operations on Sets. Use \in for “is an element of”. E.g, letting \mathbb{P} be the set of primes, then, $5 \in \mathbb{P}$ yet $6 \notin \mathbb{P}$. Changing the emphasis, $\mathbb{P} \ni 5$ [“ \mathbb{P} owns 5”] yet $\mathbb{P} \not\ni 6$ [“ \mathbb{P} does-not-own 6”]

For subsets A and B of the same space, Ω , the *inclusion relation* $A \subset B$ means:

$$\forall \omega \in A, \text{ necessarily } B \ni \omega.$$

And this can be written $B \supset A$. Use $A \subsetneq B$ for proper inclusion, i.e, $A \subset B$ yet $A \neq B$.

The *difference set* $B \setminus A$ is $\{\omega \in B \mid \omega \notin A\}$. Employ A^c for the *complement* $\Omega \setminus A$. Use $A \Delta B$ for *symmetric difference* $[A \setminus B] \cup [B \setminus A]$. Furthermore

$$A \sqcap B, \quad \text{Sets } A \text{ \& } B \text{ have at least one point in common; they intersect.}$$

$$A \sqcap B, \quad \text{The sets have no common point; disjoint.}$$

The symbol “ $A \sqcap B$ ” both asserts intersection and represents the set $A \cap B$. For a collection $\mathcal{C} = \{E_j\}_j$ of sets in Ω , let the *disjoint union* $\sqcup_j E_j$ or $\sqcup(\mathcal{C})$ represent the union $\cup_j E_j$ and also asserts that the sets are pairwise disjoint.

See next page...

SeLo [2019t] quizzes so far...

Q1: Fri. 06 Sep $[\sqrt{3}^{\sqrt{8}}]^{\sqrt{2}} = \underline{\hspace{2cm}}$. $\log_{64}(16) = \underline{\hspace{2cm}}$.

Q2: Mon. 09 Sep May lightning ⚡ strike this table! (I.e, please fill in.)

n	r_n	q_n	s_n	t_n
0	100	—	1	0
1	23		0	1
2				
3				
4				
5				

So $\text{GCD}(100, 23) = \underline{\hspace{1cm}} = [\underline{\hspace{1cm}} \cdot 100] + [\underline{\hspace{1cm}} \cdot 23]$.

Q3: Wed. 11 Sep LBolt gives $G := \text{GCD}(413, 294) = \underline{\hspace{2cm}}$. And $413S + 294T = G$, where $S = \underline{\hspace{2cm}}$ & $T = \underline{\hspace{2cm}}$ are integers.

Q4: Fri. 13 Sep Binomial coefficient $\binom{7}{4} = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$.

Q5: Mon. 16 Sep Multinomial coefficient $\binom{9}{4, 2, 3} = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$.

[Note: Write your ans. ITOF factorials, then **also** write it as a single integer, or product of two, **without** factorials.]

Q6: Wed. 18 Sep Write the free vars in each of these expressions.

$$\exists n \in \mathbb{N}: f(n) \subset \underbrace{\bigcup_{\ell=p-4}^{p+7}}_{E2} \underbrace{\{x \in \mathbb{Z} \mid \ell \cdot n \equiv_5 x^2\}}_{E1}$$

E3

$E3: \underline{\hspace{2cm}}$. $E2: \underline{\hspace{2cm}}$. $E1: \underline{\hspace{2cm}}$.

Q7: ^{Fri.}_{20 Sep} Write the truth-table for $B \Rightarrow [[\neg B] \Rightarrow C]$.

B	C	$\neg B$	$[\neg B] \Rightarrow C$	$B \Rightarrow [[\neg B] \Rightarrow C]$
T	T			
T	F			
F	T			
F	F			

Q8: ^{Wed.}_{02 Oct} [In class, we covered the definition of *convex set*.]

Do one of (a) or (b)

a Sets $A, B \subset \mathbb{R} \times \mathbb{R}$ are convex. Prove $A \cap B$ is convex.

b *Am I in class today?*

circle one *“Yes!”* *“Of course!”*

Q9: ^{Fri.}_{18 Oct} Magic integers $G_1 = \underline{\hspace{2cm}}$, $G_2 = \underline{\hspace{2cm}}$, $G_3 = \underline{\hspace{2cm}}$, each in $(-165..165]$, are st. mapping $g: \mathbb{Z}_6 \times \mathbb{Z}_5 \times \mathbb{Z}_{11} \rightarrow \mathbb{Z}_{330}$ is a ring-isomorphism, where

$$g((z_1, z_2, z_3)) := \langle z_1 G_1 + z_2 G_2 + z_3 G_3 \rangle_{330}.$$

Verify for your map: $g((1, 1, 1)) = 1$ and $[5 \cdot 11] \mid G_1$ and analogously for G_2 and G_3 .

QA: ^{Fri.}_{22 Nov} Both \sim and \bowtie are equiv-relations on a set Ω .

Define binrels \mathbf{I} and \mathbf{U} on Ω as follows.

Define $\omega \mathbf{U} \lambda$ IFF Either $\omega \sim \lambda$ or $\omega \bowtie \lambda$ [or both].

Define $\omega \mathbf{I} \lambda$ IFF Both $\omega \sim \lambda$ and $\omega \bowtie \lambda$.

So “ \mathbf{U} is an equiv-relation” is: T F

So “ \mathbf{I} is an equiv-relation” is: T F

Let \mathcal{P}_∞ denote the family of all *co-finite* subsets of \mathbb{N} . That is, a subset $S \subset \mathbb{N}$ is an *element* of \mathcal{P}_∞ IFF $\mathbb{N} \setminus S$ is finite. Define relation \approx^* on \mathcal{P}_∞ by: $A \approx^* B$ IFF $A \cap B$ is infinite.

Stmnt “This \approx^* is an equivalence-relation” is: T F

QB: ^{Mon.}_{25 Nov} ?? Solve some of the World’s Problems.