Number Sets. An expression such as $k \in \mathbb{N}$ (read as "k is an element of $\mathbb{N}$" or "k in $\mathbb{N}$") means that k is a natural number; a ***natnum***.

$\mathbb{N}$ = natural numbers = $\{0, 1, 2, \dots\}$.

$\mathbb{Z}$ = integers = $\{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the ***posints***, use $\mathbb{Z}_+$. Use $\mathbb{Z}_-$ for the negative integers, the ***negints***.

$\mathbb{Q}$ = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use $\mathbb{Q}_+$ for the positive ***ratnums*** and $\mathbb{Q}_-$ for the negative ratnums.

$\mathbb{R}$ = reals. The ***posreals*** $\mathbb{R}_+$ and the ***negreals*** $\mathbb{R}_-$.

$\mathbb{C}$ = complex numbers, also called the ***complexes***.

For $\omega \in \mathbb{C}$, let "$\omega > 5$" mean "ω *is* ***real*** and $\omega > 5$". [Use the same convention for $\geqslant, <, \leqslant$, and also if 5 is replaced by any real number.]

An "***interval of integers***" $[b \,..\, c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[\mathrm{e} \,..\, 2\pi] = \{3, 4, 5, 6\} = [3 \,..\, 6] = (2 \,..\, 6]$. We allow b and c to be $\pm\infty$; so $(-\infty \,..\, -1]$ is $\mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2i| = \sqrt{29}$.

Mathematical objects. Seq: '*sequence*'. poly(s): '*polynomial(s)*'. irred: '*irreducible*'. Coeff: '*coefficient*' and var(s): '*variable(s)*' and parm(s): '*parameter(s)*'. Expr.: '*expression*'. Fnc: '*function*' (so ratfnc: means rational function, a ratio of polynomials). cty: '*continuity*'. cts: '*continuous*'. diff'able: '*differentiable*'. CoV: '*Change-of-Variable*'. CoI: '*Constant of Integration*'. LoI: '*Limit(s) of Integration*'.

Soln: '*Solution*'. Thm: '*Theorem*'. Prop'n: '*Proposition*'. CEX: '*Counterexample*'. eqn: '*equation*'. RhS: '*RightHand Side*' of an eqn or inequality. LhS: '*left-hand side*'. Sqrt or Sqroot: '*square-root*', e.g, "the sqroot of 16 is 4". Ptn: '*partition*', but pt: '*point*', as in "a fixed-pt of a map".

FTC: '*Fund. Thm of Calculus*'. IVT: '*intermediate-Value Thm*'. MVT: '*Mean-Value Thm*'.

The ***logarithm*** fnc, defined for $x>0$, is $\log(x) := \int_1^x \frac{dv}{v}$. Its inverse-fnc is $\exp()$. For $x>0$, then, $\exp(\log(x)) = x = \mathrm{e}^{\log(x)}$. For real t, naturally, $\log(\exp(t)) = t = \log(\mathrm{e}^t)$. PolyExp: '*Polynomial-times-exponential*'. E.g, $F(t) := [3 + t^2] \cdot \mathrm{e}^{4t}$ is a polyExp.

Prefix nt- means '*non-trivial*'. E.g "a *nt*-soln to $f' = 5f$ is $f(t) := \mathrm{e}^{5t}$; a *trivial* soln is $f \equiv 0$."

Phrases. WLOG: '*Without loss of generality*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. Use iff: '*if and only if*'.

IST: '*It Suffices to*' as in ISTShow, ISTExhibit.

Use w.r.t: '*with respect to*' and s.t: '*such that*'.

Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. QED: *quod erat demonstrandum*, meaning "end of proof".

Factorial. Def: $n! := n \cdot [n-1] \cdot [n-2] \cdots 2 \cdot 1$; so $0! = 1$.

Rising Fctrl: $[\![x \uparrow K]\!] := x \cdot [x+1] \cdot [x+2] \cdots [x + [K-1]]$,

Falling Fctrl: $[\![x \downarrow K]\!] := x \cdot [x-1] \cdot [x-2] \cdots [x - [K-1]]$,

for natnum K and $x \in \mathbb{C}$. E.g, $[\![K \downarrow K]\!] = K! = [\![1 \uparrow K]\!]$.

N.B: For $n \in \mathbb{Z}$: If $K > n$ then $[\![n \downarrow K]\!] = 0$.

Note $[\![x \uparrow K]\!] = [\![x + [K-1] \downarrow K]\!]$.

Sample questions

Q: Wed. 27Sep The solutions to $3x^2 = 2 - 2x$ are $x =$ ______.

Q: Wed. 27Sep $\left[\left[\sqrt[3]{2}\right]^{\sqrt{2}}\right]^{\sqrt{8}} =$ ______. $\log_8(4) =$ ______.

DfyQ [2018t] quizzes so far...

Q1: Wed. 29 Aug $[\sqrt{3}^{\sqrt{8}}]^{\sqrt{2}} =$ 9. $\log_{64}(16) =$ 2/3.

Q2: Fri. 31 Aug U.F $y = y(t)$ satisfies $y'' + 3y' + 2y = 0$, with $y(0) = 2$ and $y'(0) = 11$. So $y(t) = A\mathrm{e}^{\alpha t} + B\mathrm{e}^{\beta t}$ where

$\alpha =$ ______, $\beta =$ ______, $A =$ ______, $B =$ ______.

ZeroTar Soln: The operator's aux-poly is

$$q(z) := z^2 + 3z + 2 = [z - -1] \cdot [z - -2],$$

so $\alpha = -1$ and $\beta = -2$, or *vice versa*. The init.conds yield $\boxed{2 = A + B}$ and $\boxed{11 = \alpha A + \beta B = -A - 2B}$. Adding, gives $B = -13$, whence $A = 15$. Consequently, the IVP soln is $y(t) = 15\mathrm{e}^{-t} - 13\mathrm{e}^{-2t}$.

Q3: Wed. 05 Sep Blanks$\in\mathbb{R}$. So $\frac{1}{2+3\boldsymbol{i}} = \underline{\qquad} + \boldsymbol{i}\cdot\left[\underline{\qquad}\right]$.

Thus $\operatorname{Im}\left(\frac{5-\boldsymbol{i}}{2+3\boldsymbol{i}}\right) = \underline{\qquad}$.

By the way, $|5-3\boldsymbol{i}| = \underline{\qquad}$.

Soln: $\mathbb{C}$-arithmetic. With $d := 2+3\boldsymbol{i}$, note

$$\frac{1}{d} \;=\; \frac{1}{d}\cdot\frac{\overline{d}}{\overline{d}} \;=\; \overline{d}\big/|d|^2 .$$

Since $|d|^2 = 2^2+3^2 = 13$, we have that

$$\frac{1}{d} \;=\; \frac{\overline{d}}{13} \;=\; \frac{2}{13} + \frac{-3}{13}\boldsymbol{i} .$$

Letting $n := 5-\boldsymbol{i}$, the above shows that

$$\frac{n}{d} \;=\; \frac{n\cdot\overline{d}}{|d|^2} \;=\; \frac{n\cdot\overline{d}}{13} .$$

Hence

$$\operatorname{Im}\left(\frac{n}{d}\right) \;=\; \operatorname{Im}\left(n\cdot\overline{d}\right)/13 ,$$

since $13\in\mathbb{R}$. Finally,

$$\operatorname{Im}\left(n\cdot\overline{d}\right) \;=\; 5\cdot[-3] + [-1]\cdot 2 \;=\; -17 .$$

Thus

$$\operatorname{Im}\left(\frac{n}{d}\right) \;=\; \frac{-17}{13} .$$

Lastly, $|5-3\boldsymbol{i}|^2 = 5^2+3^2 = 34$, so $|5-3\boldsymbol{i}| = \sqrt{34}$.

Q4: Wed. 12 Sep Value $\operatorname{Im}\left(\frac{1}{4-3\boldsymbol{i}}\right) = \underline{\frac{3}{4^2+3^2} = \frac{3}{25}}$.

Suppose $[C + D\boldsymbol{i}]^2 = -4\boldsymbol{i}$, where $C,D\in\mathbb{R}$. Then $C = \underline{\pm\sqrt{2}}$ and $D = \underline{\mp\sqrt{2}}$.

Q5: Fri. 14 Sep For $t>0$, fnc $y_\alpha(t) := \underline{\qquad}$ is the gen.soln to $y' + \left[\frac{2}{t}\cdot y\right] = t^3$. [*Hint:* FOLDE. The coefficient and target fncs are $C(t) := \frac{2}{t}$ and $G(t) := t^3$.]

Soln: Using our FOLDE notation, the coeff and tar fncs are $C(t) := \frac{2}{t}$ and $G(t) := t^3$. Whence $B := \int C$ is

$$B(t) \;=\; 2\log(t) . \quad \textit{Hence, } M(t) := e^{2\log(t)} \;=\; t^2 .$$

So $P(t) := t^2\cdot G(t) = t^5$ and $Q(t) := \int^t P = \frac{1}{6}t^6$. Thus

$$y_\alpha(t) \;=\; \frac{\alpha+Q(t)}{M(t)} \;\overset{\text{note}}{=\!=}\; \frac{\alpha}{t^2} + \frac{1}{6}t^4$$

is our general soln to the given DE. [Did you check?]

Q6: Mon. 17 Sep DE $\left[\mathcal{N}(x,y)\cdot\frac{dy}{dx}\right] + \mathcal{M}(x,y) = 0$ is *exact*, where

$\mathcal{N}(x,y) := \left[x^2-7\right]$ and $\mathcal{M}(x,y) := 2xy + 3e^{3x}$.

Its solns $y=y(x)$ satisfy $\mathbf{F}(x,y(x)) = \text{Const}$, where

$\mathbf{F}(x,y) = \underline{\qquad}$.

Exact DE. Computing, $\mathcal{N}_x = 2x$ and $\mathcal{M}_y = 2x+0$, so our DE is indeed exact. Anti-diffing,

$$\mathcal{B} := \int^y \mathcal{N} = x^2y - 7y \quad\text{and}$$

$$\mathcal{A} := \int^x \mathcal{M} = x^2y + e^{3x}. \quad\text{Hence,}$$

$$\mathcal{B} - \mathcal{A} = \underbrace{-7y}_{g(y)} - \underbrace{e^{3x}}_{h(x)}. \quad\textit{Thus,}$$

$$\mathbf{F}(x,y) \;=\; \mathcal{A}+g \;=\; x^2y + e^{3x} - 7y .$$

Reassuringly, $\mathbf{F}$ also equals $\mathcal{B}+h$. Checking, verify that $\mathbf{F}_y = \mathcal{N}$ and $\mathbf{F}_x = \mathcal{M}$.

Q7: Fri. 21 Sep *A* particular soln $p=p(t)$ to

$$p' + 2p \;=\; 6t^2 + 8t + 1$$

is $p(t) = \underline{\qquad}\cdot t^2 + \underline{\qquad}\cdot t + \underline{\qquad}$.

The general soln is $y_\alpha(t) = \alpha e^{Mt} + p(t)$, where $M = \underline{\qquad}$. [Put correct numbers in the four blanks.]

Answer: Using the Polynomial target section of DiffyNotes, a poly soln $p=p(t)$ has form

$$p \;=\; wt^2 + vt + u . \quad \textit{Hence,}$$

$$p' + 2p \;=\; 2wt^2 + [2w+2v]t + [u+2] .$$

Setting this last equal to $6t^2+8t+1$ hands us eqns, $2w=6$ and $2w+2v=8$ and $v+2u=1$. Reading L-to-R, then, $w=3$ and $v=1$ and $u=0$.

So polynomial $p(t) = 3t^2 + t + 0$; satisfies the DE. The *gen.soln* is $y_\alpha(t) = \alpha e^{-2t} + 3t^2 + t$.

Q8: Mon. 24 Sep The DOp in $y' - 2y \stackrel{*}{=} [4t+3]e^{3t}$ is $\mathsf{L} := \mathbf{D} - 2\mathbf{I}$.

It's *associated op* $\mathsf{V}{=}\mathsf{V}_{\mathsf{L},3}$ satisfies $\mathsf{L}(fe^{3t}) = \mathsf{V}(f)\,e^{3t}$, where $\mathsf{V}(f) =$ __________. Thus $y := G(t)e^{3t}$ satisfies $(*)$, for *polynomial* $G(t) =$ __________.

Soln: Let $E := e^{3t}$. Then

$$\mathsf{L}(fE) = [f' + 3f]E - 2fE = [f' + f]E;$$

so $\mathsf{V}(f) = f' + f$. For w,v not-yet-determined coeffs,

$$4t + 3 = \mathsf{V}(wt + v) \stackrel{\text{note}}{=\!=} wt + [w + v].$$

Thus $w = 4$ and $v = \text{-}1$. I.e, $G(t) = 4t - 1$.

Q9: Fri. 05 Oct **A** soln (use PolyExp) to $y'' + y = [4t - 2]e^t$ is $y(t) =$ $[2t - 3]e^t$.

Soln: Use $G(t) := 4t - 2$ for the target polynomial. Let $E := e^t$; so $E' = E$. What must a fnc f satisfy, so that $[\mathbf{D}^2 + \mathbf{I}](f{\cdot}E) = G{\cdot}E$? Computing, $[\mathbf{D}^2 + \mathbf{I}](fE)$ equals

$$[f''E + 2f'E' + fE''] + fE \stackrel{\text{note}}{=\!=} [\overbrace{f'' + 2f' + 2f}^{=\mathsf{V}(f)}]{\cdot}E.$$

Let's specialize to constructing a *polynomial*, h, s.t $[\mathbf{D}^2 + \mathbf{I}](h{\cdot}E) = G{\cdot}E$. Thus we seek numbers A, B so that $h(t) := \mathrm{A}t + \mathrm{B}$ satisfies $\mathsf{V}(h) = G$. Note that $\mathsf{V}(h) = 2[h' + h]$. (*Why?*) So $G/2$ equals

$$2t - 1 = h' + h \stackrel{\text{note}}{=\!=} \mathrm{A}t + [\mathrm{A} + \mathrm{B}].$$

Reading L-to-R, $\mathrm{A} = 2$ hence $\mathrm{B} = \text{-}3$.

CODA: Gen-soln is $y_{\alpha,\beta}(t) = [2t - 3]e^t + \alpha e^{\boldsymbol{i}t} + \beta e^{-\boldsymbol{i}t}$.

NOTE: A few past students have added a term to their PolyExp answer, writing $+\alpha e^{-t}$ or $+\alpha e^{t}$ or $+\alpha$. Minorly, this ignores that the problem asks for just *one* soln.

Majorly, all of these render a potentially correct soln, incorrect. For their exponent-multipliers are -1, 1 and 0, respectively. But none of these are roots of the aux-poly of $\mathbf{D}^2 + \mathbf{I}$, the original operator. Its aux-poly factors as $[z - \boldsymbol{i}]{\cdot}[z + \boldsymbol{i}]$, with roots $\pm\boldsymbol{i}$.

QA: Mon. 08 Oct For CCLDOp $\mathsf{L} := \mathbf{D}^3 - 3\mathbf{D} + 2\mathbf{I}$ and thrice diff'able fnc f, note $\mathsf{L}(f{\cdot}e^{2t}) = \mathsf{V}(f){\cdot}e^{2t}$, where CCLDOp V is

$\mathsf{V} =$ ____ $\mathbf{D}^3 +$ ____ $\mathbf{D}^2 +$ ____ $\mathbf{D} +$ ____ $\mathbf{I}$.

[Put the correct number in *each* of the four blanks; zero, one, fractions, and negative numbers are allowed.]

Soln: With $E := e^{2t}$, note $E' = 2E$. Thus

$$\begin{aligned}
\mathbf{I}(f{\cdot}E) &= fE, \qquad \textit{and} \\
\mathbf{D}(f{\cdot}E) &= f'E + fE' = [f' + 2f]E. \qquad \textit{Continuing,} \\
\mathbf{D}^2(f{\cdot}E) &= f''E + 2f'E' + fE'' = [f'' + 2{\cdot}2f' + 4f]E. \quad \textit{Also,} \\
\mathbf{D}^3(f{\cdot}E) &= f'''E + 3f''E' + 3f'E'' + fE''' \\
&= [f''' + 3{\cdot}2f'' + 3{\cdot}4f' + 8f]E.
\end{aligned}$$

Combining these according to L says that $\mathsf{V}(f)$ equals

$$[f''' + 3{\cdot}2f'' + 3{\cdot}4f' + 8f] - 3[f' + 2f] + 2f.$$

Computing, $\mathsf{V}(f) = f''' + 6f'' + 9f' + 4f$.
Consequently, $\mathsf{V} = 1\mathbf{D}^3 + 6\mathbf{D}^2 + 9\mathbf{D} + 4\mathbf{I}$.

QB: ?? Wed. 10 Oct Solve some of the World's Problems.