

Number Sets. An expression such as $k \in \mathbb{N}$ (read as “ k is an element of \mathbb{N} ” or “ k in \mathbb{N} ”) means that k is a natural number; a *natnum*.

\mathbb{N} = natural numbers = $\{0, 1, 2, \dots\}$.

\mathbb{Z} = integers = $\{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the *posints*, use \mathbb{Z}_+ . Use \mathbb{Z}_- for the negative integers, the *negints*.

\mathbb{Q} = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use \mathbb{Q}_+ for the positive *ratnums* and \mathbb{Q}_- for the negative ratnums.

\mathbb{R} = reals. The *posreals* \mathbb{R}_+ and the *negreals* \mathbb{R}_- .

\mathbb{C} = complex numbers, also called the *complexes*.

An “*interval of integers*” $[b..c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[e..2\pi] = \{3, 4, 5, 6\} = [3..6] = (2..6]$. We allow b and c to be $\pm\infty$; so $(-\infty..-1]$ is \mathbb{Z}_- .

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$.
Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2i| = \sqrt{29}$.

Mathematical objects. Seq: ‘sequence’. poly(s): ‘polynomial(s)’. irred: ‘irreducible’. Coeff: ‘coefficient’ and var(s): ‘variable(s)’ and parm(s): ‘parameter(s)’. Expr.: ‘expression’. Fnc: ‘function’ (so ratfnc: means rational function, a ratio of polynomials). cty: ‘continuity’. cts: ‘continuous’. diff’able: ‘differentiable’. CoV: ‘Change-of-Variable’. Col: ‘Constant of Integration’. Lol: ‘Limit(s) of Integration’.

Soln: ‘Solution’. Thm: ‘Theorem’. Prop’n: ‘Proposition’. CEX: ‘Counterexample’. eqn: ‘equation’. RhS: ‘RightHand Side’ of an eqn or inequality. LhS: ‘left-hand side’. Sqrt or Sroot: ‘square-root’, e.g, “the sroot of 16 is 4”. Ptn: ‘partition’, but pt: ‘point’, as in “a fixed-pt of a map”.

FTC: ‘Fund. Thm of Calculus’. IVT: ‘intermediate-Value Thm’. MVT: ‘Mean-Value Thm’.

The *logarithm* fnc, defined for $x > 0$, is $\log(x) := \int_1^x \frac{dv}{v}$. Its inverse-fnc is $\exp()$. For $x > 0$, then, $\exp(\log(x)) = x = e^{\log(x)}$. For real t , naturally, $\log(\exp(t)) = t = \log(e^t)$. PolyExp: ‘Polynomial-times-exponential’. E.g, $F(t) := [3 + t^2] \cdot e^{4t}$ is a polyExp.

Phrases. WLOG: ‘Without loss of generality’. TFAE: ‘The following are equivalent’. ITOF: ‘In Terms Of’. OTForm: ‘of the form’. FTSOC: ‘For the sake of contradiction’. Use iff: ‘if and only if’.

IST: ‘It Suffices to’ as in ISTShow, ISTExhibit.

Use w.r.t: ‘with respect to’ and s.t: ‘such that’.

Latin: e.g: *exempli gratia*, ‘for example’. i.e: *id est*, ‘that is’. N.B: *Nota bene*, ‘Note well’. QED: *quod erat demonstrandum*, meaning “end of proof”.

Factorial. Def: $n! := n \cdot [n-1] \cdot [n-2] \cdots 2 \cdot 1$; so $0! = 1$.

Rising Fctrl: $\llbracket x \uparrow K \rrbracket := x \cdot [x+1] \cdot [x+2] \cdots [x+[K-1]]$,

Falling Fctrl: $\llbracket x \downarrow K \rrbracket := x \cdot [x-1] \cdot [x-2] \cdots [x-[K-1]]$,

for natnum K and $x \in \mathbb{C}$. E.g, $\llbracket K \downarrow K \rrbracket = K! = \llbracket 1 \uparrow K \rrbracket$.

N.B: For $n \in \mathbb{Z}$: If $K > n$ then $\llbracket n \downarrow K \rrbracket = 0$.

Note $\llbracket x \uparrow K \rrbracket = \llbracket x + [K-1] \downarrow K \rrbracket$.

Summation function. Given a function f on \mathbb{N} , its *summation function* is

$$1a: \quad \widehat{f}(N) := \sum_{n \in [0..N)} f(n).$$

If f is a polynomial of degree $D \in \mathbb{N}$, then \widehat{f} is a polynomial of degree $D+1$.

To see this, define the K^{th} *binomial polynomial*, for $K \in \mathbb{N}$, by

$$1b: \quad \mathcal{B}_K(x) := \frac{x \cdot [x-1] \cdot [x-2] \cdots [x-[K-1]]}{K!},$$

which we may also write as $\binom{x}{K} = \frac{\llbracket x \downarrow K \rrbracket}{K!}$. Rewrite the binomial identity $\binom{n}{K+1} = \binom{n-1}{K+1} + \binom{n-1}{K}$ as

$$\binom{n-1}{K} = \binom{n}{K+1} - \binom{n-1}{K+1}.$$

Summing over n , and using that $\binom{0}{K+1} = 0$ [since $K+1$ is positive] shows that

$$1c: \quad \widehat{\mathcal{B}}_K = \mathcal{B}_{K+1}.$$

The binomial polys $\{\mathcal{B}_K\}_{K=0}^\infty$ form a basis for the vectorspace of polys. Since the $f \mapsto \widehat{f}$ map is linear, we can compute the summation-poly of arbitrary polynomials.

[ASIDE: Stronger, collection $\{\mathcal{B}_K\}_{K=0}^\infty$ is a \mathbb{Z} -basis for the set of \mathbb{Z} -coeff polynomials; however, this fact isn’t obvious.]

Combinatorial graphs. Some notation:

General graphs, G, H, D, S .

Complete graph: K_N . Complete bipartite graph: $K_{3,5}$.

Cyclic graph: C_N . Wheel graph: W_{N+1} , is a vertex attached to each of the N vertices of C_N .

Path-graph P , which is a special case of a tree-graph, T_N .

Named graph, e.g, *Empty*.

Comb quizzes so far...

P1: ^{Wed.}_{27Sep} Equality $\binom{74}{26, 25, 23} = \binom{74}{25} \cdot \binom{N}{K}$ suggests that $N =$ and $K =$

Let G_n be the number of coin-flip sequences of length $2n$, with "first-return-to-zero" happening at time $2n$. Let F_n count those such sequences that start with HEADS.

Then $F_{50} =$

FRtZero Soln: Using a REFLECTION PRINCIPLE we showed, for N a natnum, that

$$F_{N+1} = \binom{2N}{N, N} - \binom{2N}{N-1, N+1} \stackrel{\text{note}}{=} \frac{1}{N+1} \binom{2N}{N, N}.$$

This value is called the N^{th} **Catalan number**, C_N .

In our case, $N+1 = 50$, so $N = 49$. Consequently,

$$F_{50} = C_{49} = \frac{\binom{98}{49}}{50} = \binom{98}{49, 49} - \binom{98}{48, 50}.$$

ASIDE: From class, Stirling's formula, $n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$, yields the asymptotic relation $\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}$. Thus

$$C_n \sim \frac{1}{n+1} \cdot \frac{4^n}{\sqrt{\pi n}} \sim \frac{4^n}{n^{3/2} \sqrt{\pi}}, \text{ as } n \nearrow \infty.$$

Thus C_{49} is approximately $4^{49} / [7^3 \cdot \sqrt{\pi}]$.

P2: ^{Mon.}_{02Oct} The number of diagonal lattice-paths from $(0, 0)$ to $(31, 7)$ which *never* touch the $y=10$ line is

Soln. Paths from $\mathbf{A} := (0, 0)$ to $\mathbf{B} := (31, 7)$, have $7 - 0 = 7$ more Us than Ds. Thus $[7 + \#D] + \#D = 31$, so $\#D=12$. and $\#U=19$. Hence $|\text{PATH}(\mathbf{A} \rightarrow \mathbf{B})| = \binom{31}{19, 12}$.

A *bad path* touches the $y=10$ line. Reflecting \mathbf{B} across that line gives point $\mathbf{B}' := (31, 13)$, since $13 - 10 = 10 - 7$. An $\mathbf{A} \rightarrow \mathbf{B}'$ path has 13 more Us than Ds. So $\#D=9$. and $\#U=22$. Since $\text{BAD}(\mathbf{A} \rightarrow \mathbf{B})$ is in 1-to-1 correspondence with $\text{PATH}(\mathbf{A} \rightarrow \mathbf{B}')$, we have $|\text{BAD}(\mathbf{A} \rightarrow \mathbf{B})| = \binom{31}{22, 9}$.

Putting it all together, the number of good paths is

$$|\text{GOOD}(\mathbf{A} \rightarrow \mathbf{B})| = \binom{31}{19, 12} - \binom{31}{22, 9}.$$

P3: ^{Wed.}_{04Oct} Stirling's Formula says: $n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$.

Am I in class today?

circle (one) "Yes!" "Of course!"

P4: ^{Mon.}_{09Oct} The 4th Bell number is 15. List, in number-of-atoms order, the 15 partitions of $\{1, 2, 3, 4\}$.

Ptns: Contiguous symbols make up an atom.

[1234]	[1, 234]	[1, 2, 34]	[1, 2, 3, 4]
	[14, 23]	[1, 24, 3]	
	[13, 24]	[14, 2, 3]	
	[134, 2]	[1, 23, 4]	
	[12, 34]	[13, 2, 4]	
	[124, 3]	[12, 3, 4]	
	[123, 4]		

And

$$1 + 7 + 6 + 1 \text{ equals } \mathcal{S}(4, 1) + \mathcal{S}(4, 2) + \mathcal{S}(4, 3) + \mathcal{S}(4, 4) \stackrel{\text{of course}}{=} 15.$$

P5: ^{Wed.}_{11Oct} Permutation β has cycle-signature $[5^9]$.

It has 1 many 4th-roots with sig $[5^9]$,

and $\frac{1}{2} \cdot \binom{9}{4, 4, 1} \cdot [15 \cdot 10 \cdot 5]^2$ with sig $[20^2, 5^1]$.

[Answers can be a product of multinomials, powers, numbers.]

Finding your Roots: Taking the K^{th} -power of an N -cycle breaks the N -cycle into $c := \text{Gcd}(N, K)$ many ℓ -cycles, where $\ell := \frac{N}{c}$.

Since $4 \perp 5$, a 4th-power of a 5-cycle is a 5-cycle. So β has only one $[5^9]$ -4th-root; namely β^{-1} . [DYN Notice that $\beta^5 = Id$, whence $\beta = [\beta^{-1}]^4$?]

The 4th-power of a 20-cycle has four 5-cycles. The number of ways of partitioning the nine 5-cycles into *Four,Four,One* is

$$G := \frac{\binom{9}{4,4,1}}{2! \cdot 1!} = \frac{630}{2} = 315.$$

How many ways are there to intercalate four 5-cycles to make a 20-cycle? Fix a token, *A*, of one 5-cycle. There are 20 - 5 = 15 tokens that can follow *A* in the 20-cycle; pick one, call it *B*. To follow *B* [in the 20-cycle] there are 10 tokens; pick one, *C*. To follow *C*, pick one of the 5 remaining tokens. Whence $[15 \cdot 10 \cdot 5] = 750$ choices. So $G \cdot [750]^2 = 2^2 3^4 5^7 7 = 177,187,500$ many $[20^2, 5^1]$ -4th-roots.

ADDENDUM: Alternatively, there are $[4 - 1]!$ cyclic-orderings of the four 5-cycles. And there are $5^{[4-1]}$ ways of twisting, relative to one of them. Unsurprisingly, $3! \cdot 5^3$ equals $[15 \cdot 10 \cdot 5]$.

P6: Wed. 25Oct *The Broken Problem:* Handed-in = 30pts.

Permutation β has cycle-signature $[5^9]$.

The # of β -5th-roots with sig $[20^1, 15^1, 5^2]$ is:

Oy. The 5th-power of $[20^1, 15^1, 5^2]$ has five 3-cycles [from the 15¹] hence cannot equal β . The 4th-power of $[20^1, 15^1, 5^2]$ has a 15-cycle, so is not β . I.e. *With sig $[20^1, 15^1, 5^2]$, our β has zero 5th-roots and zero 4th-roots. Can I go home, now?*

P7: Fri. 27Oct Bipartite graph $K_{7,2}$ is Eulerian. \textcircled{T} F

Eulerian: The girls, $\{\alpha, \beta\}$, each have vertexDeg=7; odd. The seven boys each have vertexDeg=2. There are exactly *two* OddDeg vertices, the graph is *connected and finite*. Therefore $K_{7,2}$ admits an Eulerian path/trail, but *not* an Eulerian circuit. A particular *E.trail* is: α -Abe- β -Bob- α -Carl- β -Dave- α -Ed- β -Fred- α -Glen- β .

The number of permutations of $[1..6]$ which have *neither* $\zeta(4\ 1)$ nor $\zeta(6\ 3\ 2)$ is

[Use Incl-Excl form: $\square - \square + \square - \square + \dots$ as appropriate.]

Incl-Excl. Cycle $\zeta(4\ 1)$ leaves $6 - 2 = 4$ tokens with no constraints, so there are $4!$ "bad" perms possessing $\zeta(4\ 1)$. Similarly, $\zeta(6\ 3\ 2)$ occurs in $[6 - 3]! = 3!$ perms. Together, cycles $\zeta(4\ 1)$ and $\zeta(6\ 3\ 2)$ leave only token 5, so 5 must be mapped to itself; there is only 1 perm possessing both cycles. Thus, the number of "good" perms is

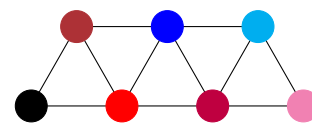
$$6! - [4! + 3!] + 1 \stackrel{\text{note}}{=} 691.$$

PBonus: Fri. 17Nov The chromatic-poly of a tree T with 5 vertices

is $\mathcal{P}_T(x) = x[x - 1]^4$

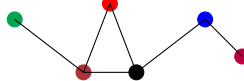
In $\Omega := [1..100]$, the number of elements which are a multiple of 3 or 5 (or both) is $\lfloor \frac{100}{3} \rfloor + \lfloor \frac{100}{5} \rfloor - \lfloor \frac{100}{15} \rfloor$ which equals $33 + 20 - 6 = 47$.

P8: Mon. 13Nov



Truss exercise: Graph has Chromatic poly $\mathcal{P}(x) = x \cdot [x - 1] \cdot [x - 2]^5$ [Can be done by inspection. Express in *chromatic form*.]

P9: Mon. 27Nov *Bird with a broken wing:* Graph



has Chromatic poly $\mathcal{P}(x) = x \cdot [x - 1]^4 \cdot [x - 2] = x^6 - 6x^5 + 14x^4 - 16x^3 + 9x^2 - 2x$. [Can be done by inspection. Do not bother to multiply out.]

PA: ^{Wed.}_{29Nov} The number of labeled 6-vertex graphs of type shown on blackboard, is:

- A:
- B:
- C:


A six of sixes: Cayley's Formula tells us to expect $6^{6-2} = 6^4 = 1296$ labeled trees on six vertices.

A:  #labelings: $6!/2 = 360$.

B:  #labelings: $6!/2 = 360$.

C:  #labelings: $6!/2 = 360$.

D:  #labelings: $6!/3! = 120$.

E:  #labelings: $6!/[2 \cdot 2 \cdot 2] = 90$.

F:  #labelings: $6!/5! = 6$.

Happily, $360 + 360 + 360 + 120 + 90 + 6 = 1296$.

PA: ^{??}_{Fri.}_{01Dec} *Chromatics? Spanning trees?*

Ameliorate some of the World's Problems.