

Number Sets. An expression such as $k \in \mathbb{N}$ (read as “ k is an element of \mathbb{N} ” or “ k in \mathbb{N} ”) means that k is a natural number; a *natnum*.

\mathbb{N} = natural numbers = $\{0, 1, 2, \dots\}$.

\mathbb{Z} = integers = $\{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the *posints*, use \mathbb{Z}_+ . Use \mathbb{Z}_- for the negative integers, the *negints*.

\mathbb{Q} = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use \mathbb{Q}_+ for the positive *ratnums* and \mathbb{Q}_- for the negative ratnums.

\mathbb{R} = reals. The *posreals* \mathbb{R}_+ and the *negreals* \mathbb{R}_- .

\mathbb{C} = complex numbers, also called the *complexes*.

An “*interval of integers*” $[b..c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[e..2\pi] = \{3, 4, 5, 6\} = [3..6] = (2..6]$. We allow b and c to be $\pm\infty$; so $(-\infty..-1]$ is \mathbb{Z}_- .

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$.
Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$
and $|-5 + 2i| = \sqrt{29}$.

Mathematical objects. Seq: ‘sequence’. poly(s): ‘polynomial(s)’. irred: ‘irreducible’. Coeff: ‘coefficient’ and var(s): ‘variable(s)’ and parm(s): ‘parameter(s)’. Expr.: ‘expression’. Fnc: ‘function’ (so ratfnc: means rational function, a ratio of polynomials). cty: ‘continuity’. cts: ‘continuous’. diff’able: ‘differentiable’. CoV: ‘Change-of-Variable’. Col: ‘Constant of Integration’. Lol: ‘Limit(s) of Integration’.

Soln: ‘Solution’. Thm: ‘Theorem’. Prop’n: ‘Proposition’. CEX: ‘Counterexample’. eqn: ‘equation’. RhS: ‘RightHand Side’ of an eqn or inequality. LhS: ‘left-hand side’. Sqrt or Sroot: ‘square-root’, e.g, “the sroot of 16 is 4”. Ptn: ‘partition’, but pt: ‘point’, as in “a fixed-pt of a map”.

FTC: ‘Fund. Thm of Calculus’. IVT: ‘intermediate-Value Thm’. MVT: ‘Mean-Value Thm’.

The *logarithm* fnc, defined for $x > 0$, is $\log(x) := \int_1^x \frac{dv}{v}$. Its inverse-fnc is $\exp()$. For $x > 0$, then, $\exp(\log(x)) = x = e^{\log(x)}$. For real t , naturally, $\log(\exp(t)) = t = \log(e^t)$. PolyExp: ‘Polynomial-times-exponential’. E.g, $F(t) := [3 + t^2] \cdot e^{4t}$ is a polyExp.

Phrases. WLOG: ‘Without loss of generality’. TFAE: ‘The following are equivalent’. ITOF: ‘In Terms Of’. OTForm: ‘of the form’. FTSOC: ‘For the sake of contradiction’. Use iff: ‘if and only if’.

IST: ‘It Suffices to’ as in ISTShow, ISTExhibit.

Use w.r.t: ‘with respect to’ and s.t: ‘such that’.

Latin: e.g: *exempli gratia*, ‘for example’. i.e: *id est*, ‘that is’. N.B: *Nota bene*, ‘Note well’. QED: *quod erat demonstrandum*, meaning “end of proof”.

Factorial. Def: $n! := n \cdot [n-1] \cdot [n-2] \cdots 2 \cdot 1$; so $0! = 1$.

Rising Fctrl: $\llbracket x \uparrow K \rrbracket := x \cdot [x+1] \cdot [x+2] \cdots [x+[K-1]]$,

Falling Fctrl: $\llbracket x \downarrow K \rrbracket := x \cdot [x-1] \cdot [x-2] \cdots [x-[K-1]]$,

for natnum K and $x \in \mathbb{C}$. E.g, $\llbracket K \downarrow K \rrbracket = K! = \llbracket 1 \uparrow K \rrbracket$.

N.B: For $n \in \mathbb{Z}$: If $K > n$ then $\llbracket n \downarrow K \rrbracket = 0$.

Note $\llbracket x \uparrow K \rrbracket = \llbracket x + [K-1] \downarrow K \rrbracket$.

Summation function. Given a function f on \mathbb{N} , its *summation function* is

$$1a: \quad \widehat{f}(N) := \sum_{n \in [0..N)} f(n).$$

If f is a polynomial of degree $D \in \mathbb{N}$, then \widehat{f} is a polynomial of degree $D+1$.

To see this, define the K^{th} *binomial polynomial*, for $K \in \mathbb{N}$, by

$$1b: \quad \mathcal{B}_K(x) := \frac{x \cdot [x-1] \cdot [x-2] \cdots [x-[K-1]]}{K!},$$

which we may also write as $\binom{x}{K} = \frac{\llbracket x \downarrow K \rrbracket}{K!}$. Rewrite the binomial identity $\binom{n}{K+1} = \binom{n-1}{K+1} + \binom{n-1}{K}$ as

$$\binom{n-1}{K} = \binom{n}{K+1} - \binom{n-1}{K+1}.$$

Summing over n , and using that $\binom{0}{K+1} = 0$ [since $K+1$ is positive] shows that

$$1c: \quad \widehat{\mathcal{B}}_K = \mathcal{B}_{K+1}.$$

The binomial polys $\{\mathcal{B}_K\}_{K=0}^\infty$ form a basis for the vectorspace of polys. Since the $f \mapsto \widehat{f}$ map is linear, we can compute the summation-poly of arbitrary polynomials.

[ASIDE: Stronger, collection $\{\mathcal{B}_K\}_{K=0}^\infty$ is a \mathbb{Z} -basis for the set of \mathbb{Z} -coeff polynomials; however, this fact isn’t obvious.]

Combinatorial graphs. Some notation:

General graphs, G, H, D, S .

Complete graph: K_N . Complete bipartite graph: $K_{3,5}$.

Cyclic graph: C_N . Wheel graph: W_{N+1} , is a vertex attached to each of the N vertices of C_N .

Path-graph P , which is a special case of a tree-graph, T_N .

Named graph, e.g, *Empty*.

Comb II quizzes so far...

PB: ^{Fri.}_{19Jan} With $a_n := 1 + n^2$, its

OGF $A(x) := \sum_{n=0}^{\infty} a_n x^n =$

[Hint: xddx-trick]

OGF Soln. Applying xddx twice to $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, gives

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{[1-x]^2}, \quad \text{and}$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x+x^2}{[1-x]^3}. \quad \text{Thus,}$$

$$A(x) = \frac{1}{1-x} + \frac{x+x^2}{[1-x]^3} = \frac{1-x+2x^2}{[1-x]^3}.$$

At WolframAlpha: `GeneratingFunction[1+n^2, n, x]`

PC: ^{Wed.}_{31Jan} *Am I in class today?*

circle (one) *“Yes!”* *“Of course!”*

PD: ^{Mon.}_{05Feb} Perm $\alpha \in S_7$ is $\langle 7\ 1\ 4\ 2\ 6\ 3\ 5 \rangle$. A particular permutation β satisfying $\beta^3 = \alpha$, is

$\beta = \langle 7 \dots \rangle$

And $\text{Sgn}(\beta)$ is (circle): **+1** **-1**.

Soln: Our α comprises a single cycle. When $n \perp 7$, then, α has a unique n^{th} -root, which is itself a 7-cycle. Conversely, when $n \not\perp 7$ then α has *no* n^{th} -roots.

Note $-2 \cdot 3 \equiv 1$. Hence $\beta := \alpha^{-2} \stackrel{\text{note}}{=} \alpha^5$ is the unique cube-root of α . Hence $\beta = \langle 7\ 3\ 2\ 1\ 5\ 6\ 4 \rangle$.

Our β is an odd-length cycle, so $\text{Sgn}(\beta) = +1$.

PE: ^{Fri.}_{12Feb} The coeff of $x^7 y^{12}$

in $[5x + y^3 + 1]^{30}$ is $5^7 \cdot \binom{30}{7, 4, 19}$, since $\frac{12}{3}$ is 4, and...

[You may write in form number times multinomial-coeff. You can leave the multinomial-coeff as such, or write ITOF factorials.]

... we are extracting the coeff of $[5x]^7 \cdot [y^3]^4 \cdot 1^{30-[7+4]}$.

PF: ^{Fri.}_{02Mar} The # of (vertex-labeled)

forests on $[1..50]$ is

Oy! I'd intended to ask for the number of rooted (spanning) forests on $[1..50]$. This equals $[n+1]^{n-1} \downarrow_{n=50}$ i.e, equals 51^{49} .

PG: ^{Wed.}_{14Mar} The number of (labeled) spanning trees

on $[1..7]$ is $7^{7-2} = 7^5$

On $[1..N]$, the number of (labeled) rooted spanning forests with ≥ 2 components is $[N+1]^{N-1} - N^{N-1}$

PH: Wed.
28Mar The girls' prefs are:

$$G1 \begin{bmatrix} B2 \\ B1 \\ B5 \\ B4 \\ B3 \end{bmatrix}, \quad G2 \begin{bmatrix} B2 \\ B1 \\ B5 \\ B4 \\ B3 \end{bmatrix}, \quad G3 \begin{bmatrix} B5 \\ B2 \\ B3 \\ B4 \\ B1 \end{bmatrix}, \quad G4 \begin{bmatrix} B5 \\ B2 \\ B3 \\ B1 \\ B4 \end{bmatrix}, \quad G5 \begin{bmatrix} B2 \\ B5 \\ B4 \\ B3 \\ B1 \end{bmatrix}.$$

The boys' prefs are:

$$B1 \begin{bmatrix} G1 \\ G2 \\ G3 \\ G4 \\ G5 \end{bmatrix}, \quad B2 \begin{bmatrix} G4 \\ G5 \\ G3 \\ G2 \\ G1 \end{bmatrix}, \quad B3 \begin{bmatrix} G5 \\ G4 \\ G3 \\ G2 \\ G1 \end{bmatrix}, \quad B4 \begin{bmatrix} G3 \\ G5 \\ G1 \\ G4 \\ G2 \end{bmatrix}, \quad B5 \begin{bmatrix} G1 \\ G2 \\ G3 \\ G4 \\ G5 \end{bmatrix}.$$

Show the steps to compute a Stable-Matching, and the result, when the boys ask the girls. Ditto, when the girls ask the boys.

Stable Soln: Running some Lisp code.
(print-stable-both (Microquiz))

```

Prefs of girls: Best-to-Worst on each line.
Pref(G1) = (B2 B1 B5 B4 B3)
Pref(G2) = (B2 B1 B5 B4 B3)
Pref(G3) = (B5 B2 B3 B4 B1)
Pref(G4) = (B5 B2 B3 B1 B4)
Pref(G5) = (B2 B5 B4 B3 B1)

Prefs of boys: Best-to-Worst on each line.
Pref(B1) = (G1 G2 G3 G4 G5)
Pref(B2) = (G4 G5 G3 G2 G1)
Pref(B3) = (G5 G4 G3 G2 G1)
Pref(B4) = (G3 G5 G1 G4 G2)
Pref(B5) = (G1 G2 G3 G4 G5)
=====

The guys ask the gals.
Dance 1: One guy rejected.

Men:      Women:
B1        G1
B2        G4
B3        G5
B4        G3
B5        G2 Finding this stable matching took 1 dance.
-----

The gals ask the guys.
Dance 1: Three gals rejected.
Dance 2: Two gals rejected.
Dance 3: Two gals rejected.
Dance 4: One gal rejected.

Men:      Women:
B1        G1
B2        G4
B3        G3
B4        G5
B5        G2 Finding this stable matching took 4 dances.
-----

```

PI: ?? Fri.
30Mar Ameliorate some of the World's Problems.

Comb I quizzes

P1: ^{Wed.}_{27Sep} Let G_n be the number of coin-flip sequences of length $2n$, with “first-return-to-zero” happening at time $2n$. Let F_n count those such sequences that start with HEADS.

Then $F_{50} =$ _____

FRtZero Soln: Using a REFLECTION PRINCIPLE WE showed, for N a natnum, that

$$F_{N+1} = \binom{2N}{N, N} - \binom{2N}{N-1, N+1} \stackrel{\text{note}}{=} \frac{1}{N+1} \binom{2N}{N, N}.$$

This value is called the N^{th} **Catalan number**, C_N .
 In our case, $N+1 = 50$, so $N = 49$. Consequently,
 $F_{50} = C_{49} = \frac{\binom{98}{49}}{50} = \binom{98}{49, 49} - \binom{98}{48, 50}.$

ASIDE: From class, Stirling's formula, $n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$, yields the asymptotic relation $\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}$. Thus

$$C_n \sim \frac{1}{n+1} \cdot \frac{4^n}{\sqrt{\pi n}} \sim \frac{4^n}{n^{3/2} \sqrt{\pi}}, \text{ as } n \nearrow \infty.$$

Thus C_{49} is approximately $4^{49} / [7^3 \cdot \sqrt{\pi}]$.

P2: ^{Mon.}_{02Oct} The number of diagonal lattice-paths from $(0, 0)$ to $(31, 7)$ which *never* touch the $y=10$ line is _____

Soln. Paths from $\mathbf{A} := (0, 0)$ to $\mathbf{B} := (31, 7)$, have $7 - 0 = 7$ more Us than Ds. Thus $[7 + \#D] + \#D = 31$, so $\#D=12$. and $\#U=19$. Hence $|\text{PATH}(\mathbf{A} \rightarrow \mathbf{B})| = \binom{31}{19, 12}.$

A **bad path** touches the $y=10$ line. Reflecting \mathbf{B} across that line gives point $\mathbf{B}' := (31, 13)$, since $13 - 10 = 10 - 7$. An $\mathbf{A} \rightarrow \mathbf{B}'$ path has 13 more Us than Ds. So $\#D=9$. and $\#U=22$. Since $\text{BAD}(\mathbf{A} \rightarrow \mathbf{B})$ is in 1-to-1 correspondence with $\text{PATH}(\mathbf{A} \rightarrow \mathbf{B}')$, we have $|\text{BAD}(\mathbf{A} \rightarrow \mathbf{B})| = \binom{31}{22, 9}.$

Putting it all together, the number of good paths is

$$|\text{GOOD}(\mathbf{A} \rightarrow \mathbf{B})| = \binom{31}{19, 12} - \binom{31}{22, 9}.$$

P3: ^{Wed.}_{04Oct} Stirling's Formula says: $n! \sim \sqrt{2\pi n} \cdot \left[\frac{n}{e}\right]^n$.

Am I in class today?

circle (one) “Yes!” “Of course!”

P4: ^{Mon.}_{09Oct} The 4th Bell number is 15. List, in number-of-atoms order, the 15 partitions of $\{1, 2, 3, 4\}$.

Ptns: Contiguous symbols make up an atom.

[1234]	[1, 234]	[1, 2, 34]	[1, 2, 3, 4]
	[14, 23]	[1, 24, 3]	
	[13, 24]	[14, 2, 3]	
	[134, 2]	[1, 23, 4]	
	[12, 34]	[13, 2, 4]	
	[124, 3]	[12, 3, 4]	
	[123, 4]		

And $1 + 7 + 6 + 1$ equals $\mathcal{S}(4, 1) + \mathcal{S}(4, 2) + \mathcal{S}(4, 3) + \mathcal{S}(4, 4) \stackrel{\text{of course}}{=} 15.$

P5: ^{Wed.}_{11Oct} Permutation β has cycle-signature $[5^9]$.
 It has 1 many 4th-roots with sig $[5^9]$,
 and $\frac{1}{2} \cdot \binom{9}{4, 4, 1} \cdot [15 \cdot 10 \cdot 5]^2$ with sig $[20^2, 5^1]$.
 [Answers can be a product of multinomials, powers, numbers.]

Finding your Roots: Taking the K^{th} -power of an N -cycle breaks the N -cycle into $c := \text{Gcd}(N, K)$ many ℓ -cycles, where $\ell := \frac{N}{c}$.

Since $4 \perp 5$, a 4th-power of a 5-cycle is a 5-cycle. So β has only one $[5^9]$ -4th-root; namely β^{-1} . [DYNotice that $\beta^5 = \text{Id}$, whence $\beta = [\beta^{-1}]^4$?]

The 4th-power of a 20-cycle has four 5-cycles. The number of ways of partitioning the nine 5-cycles into *Four, Four, One* is

$$G := \frac{\binom{9}{4, 4, 1}}{2! \cdot 1!} = \frac{630}{2} = 315.$$

How many ways are there to intercalate four 5-cycles to make a 20-cycle? Fix a token, A , of one 5-cycle. There are $20 - 5 = 15$ tokens that can follow A in the 20-cycle; pick one, call it B . To follow B [in the 20-cycle] there are 10 tokens; pick one, C . To follow C , pick one of the 5 remaining tokens. Whence $[15 \cdot 10 \cdot 5] = 750$ choices. So $G \cdot [750]^2 = 2^2 3^4 5^7 7 = 177, 187, 500$ many $[20^2, 5^1]$ -4th-roots.

ADDENDUM: Alternatively, there are $[4 - 1]!$ cyclic-orderings of the four 5-cycles. And there are $5^{[4-1]}$ ways of twisting, relative to one of them. Unsurprisingly, $3! \cdot 5^3$ equals $[15 \cdot 10 \cdot 5]$.

P6: ^{Wed.}_{25Oct} *The Broken Problem:* Handed-in = 30pts.

Permutation β has cycle-signature $[5^9]$.
 The # of β -5th-roots with sig $[20^1, 15^1, 5^2]$ is: _____

Oy. The 5th-power of $[20^1, 15^1, 5^2]$ has five 3-cycles [from the 15¹] hence cannot equal β . The 4th-power of $[20^1, 15^1, 5^2]$ has a 15-cycle, so is not β . I.e. With sig $[20^1, 15^1, 5^2]$, our β has zero 5th-roots and zero 4th-roots. *Can I go home, now?*

P7: ^{Fri.}_{27Oct} Bipartite graph $K_{7,2}$ is Eulerian. $(T) F$

Eulerian: The girls, $\{\alpha, \beta\}$, each have vertexDeg=7; odd. The seven boys each have vertexDeg=2. There are exactly two OddDeg vertices, the graph is connected and finite. Therefore $K_{7,2}$ admits an Eulerian trail, but not an Eulerian circuit. A particular Eulerian-trail is:
 α -Abe- β -Bob- α -Carl- β -Dave- α -Ed- β -Fred- α -Glen- β .

The number of permutations of $[1..6]$ which have neither $(4\ 1)$ nor $(6\ 3\ 2)$ is _____
 [Use Incl-Excl form: $\square - \square + \square - \square + \dots$ as appropriate.]

Incl-Excl. Cycle $(4\ 1)$ leaves $6 - 2 = 4$ tokens with no constraints, so there are $4!$ “bad” perms possessing $(4\ 1)$. Similarly, $(6\ 3\ 2)$ occurs in $[6 - 3]! = 3!$ perms. Together, cycles $(4\ 1)$ and $(6\ 3\ 2)$ leave only token 5, so 5 must be mapped to itself; there is only 1 perm possessing both cycles. Thus, the number of “good” perms is

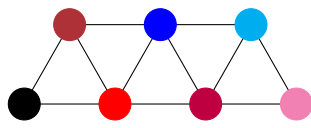
$$6! - [4! + 3!] + 1 \stackrel{\text{note}}{=} 691.$$

PBonus: ^{Fri.}_{17Nov} The chromatic-poly of a tree T with 5 vertices

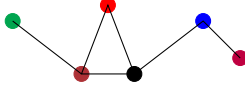
is $\mathcal{P}_T(x) = x[x - 1]^4$ _____

In $\Omega := [1..100]$, the number of elements which are a multiple of 3 or 5 (or both) is $\lfloor \frac{100}{3} \rfloor + \lfloor \frac{100}{5} \rfloor - \lfloor \frac{100}{15} \rfloor$ _____
 which equals $33 + 20 - 6 = 47$.

P8: ^{Mon.}_{13Nov}

Truss exercise: Graph 
 has Chromatic poly $\mathcal{P}(x) = x \cdot [x - 1] \cdot [x - 2]^5$ _____
 [Can be done by inspection. Express in chromatic form.]

P9: ^{Mon.}_{27Nov} *Bird with a broken wing:* Graph


 has Chromatic poly $\mathcal{P}(x) = x \cdot [x - 1]^4 \cdot [x - 2]$ _____
 $= x^6 - 6x^5 + 14x^4 - 16x^3 + 9x^2 - 2x.$
 [Can be done by inspection. Do not bother to multiply out.]

PA: ^{Wed.}_{29Nov} The number of labeled 6-vertex graphs of type shown on blackboard, is:

- A:
 B:
 C:


A six of sixes: Cayley's Formula tells us to expect $6^{6-2} = 6^4 = 1296$ labeled trees on six vertices.

A:  #labelings: $6!/2 = 360$.

B:  #labelings: $6!/2 = 360$.

C:  #labelings: $6!/2 = 360$.

D:  #labelings: $6!/3! = 120$.

E:  #labelings: $6!/[2 \cdot 2 \cdot 2] = 90$.

F:  #labelings: $6!/5! = 6$.

Happily, $360 + 360 + 360 + 120 + 90 + 6 = 1296$.