Combinatorics MAD4203 **Quizzes P** 08Dec2016

Number Sets. An expression such as $k \in \mathbb{N}$ (read as "k is an element of $\mathbb{N}$" or "k in $\mathbb{N}$") means that k is a natural number; a ***natnum***.

$\mathbb{N}$ = natural numbers = $\{0, 1, 2, \ldots\}$.

$\mathbb{Z}$ = integers = $\{\ldots, -2, -1, 0, 1, \ldots\}$. For the set $\{1, 2, 3, \ldots\}$ of positive integers, the ***posints***, use $\mathbb{Z}_+$. Use $\mathbb{Z}_-$ for the negative integers, the ***negints***.

$\mathbb{Q}$ = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+\}$. Use $\mathbb{Q}_+$ for the positive ***ratnums*** and $\mathbb{Q}_-$ for the negative ratnums.

$\mathbb{R}$ = reals. The ***posreals*** $\mathbb{R}_+$ and the ***negreals*** $\mathbb{R}_-$.

$\mathbb{C}$ = complex numbers, also called the ***complexes***.

An **"*interval of integers*"** $[b\,..\,c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[\mathrm{e}\,..\,2\pi] = \{3, 4, 5, 6\} = [3\,..\,6] = (2\,..\,6]$. We allow b and c to be $\pm\infty$; so $(-\infty\,..\,-1]$ is $\mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2i| = \sqrt{29}$.

Mathematical objects. Seq: '*sequence*'. poly(s): '*polynomial(s)*'. irred: '*irreducible*'. Coeff: '*coefficient*' and var(s): '*variable(s)*' and parm(s): '*parameter(s)*'. Expr.: '*expression*'. Fnc: '*function*' (so ratfnc: means rational function, a ratio of polynomials). cty: '*continuity*'. cts: '*continuous*'. diff'able: '*differentiable*'. CoV: '*Change-of-Variable*'. CoI: '*Constant of Integration*'. LoI: '*Limit(s) of Integration*'.

Soln: '*Solution*'. Thm: '*Theorem*'. Prop'n: '*Proposition*'. CEX: '*Counterexample*'. eqn: '*equation*'. RhS: '*RightHand Side*' of an eqn or inequality. LhS: '*lefthand side*'. Sqrt or Sqroot: '*square-root*', e.g, "the sqroot of 16 is 4". Ptn: '*partition*', but pt: '*point*', as in "a fixed-pt of a map".

FTC: '*Fund. Thm of Calculus*'. IVT: '*intermediate-Value Thm*'. MVT: '*Mean-Value Thm*'.

The ***logarithm*** fnc, defined for $x>0$, is $\log(x) := \int_1^x \frac{dv}{v}$. Its inverse-fnc is $\exp()$. For $x>0$, then, $\exp(\log(x)) = x = \mathrm{e}^{\log(x)}$. For real t, naturally, $\log(\exp(t)) = t = \log(\mathrm{e}^t)$. PolyExp: '*Polynomial-times-exponential*'. E.g, $F(t) := [3 + t^2]\cdot \mathrm{e}^{4t}$ is a polyExp.

Phrases. WLOG: '*Without loss of generality*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. Use iff: '*if and only if*'.

IST: '*It Suffices to*' as in ISTShow, ISTExhibit.

Use w.r.t: '*with respect to*' and s.t: '*such that*'.

Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. QED: *quod erat demonstrandum*, meaning "end of proof".

Factorial. Def: $n! := n\cdot[n-1]\cdot[n-2]\cdots 2\cdot 1$; so $0! = 1$.

Rising Fctrl: $[\![x \uparrow K]\!] := x\cdot[x+1]\cdot[x+2]\cdots[x+[K-1]]$,

Falling Fctrl: $[\![x \downarrow K]\!] := x\cdot[x-1]\cdot[x-2]\cdots[x-[K-1]]$,

for natnum K and $x \in \mathbb{C}$. E.g, $[\![K\downarrow K]\!] = K! = [\![1\uparrow K]\!]$.

N.B: For $n \in \mathbb{Z}$: If $K > n$ then $[\![n \downarrow K]\!] = 0$.

Note $[\![x \uparrow K]\!] = [\![x + [K-1] \downarrow K]\!]$.

Summation function. Given a function f on $\mathbb{N}$, its ***summation function*** is

1a: $$\widehat{f}(N) := \sum_{n \in [0\,..\,N)} f(n)\,.$$

If f is a polynomial of degree $D \in \mathbb{N}$, then $\widehat{f}$ is a polynomial of degree $D+1$.

To see this, define the K^{th} ***binomial polynomial***, for $K \in \mathbb{N}$, by

1b: $$\mathfrak{B}_K(x) := \frac{x\cdot[x-1]\cdot[x-2]\cdots[x-[K-1]]}{K!}\,,$$

which we may also write as $\binom{x}{K} = \frac{[\![x\downarrow K]\!]}{K!}$. Rewrite the binomial identity $\binom{n}{K+1} = \binom{n-1}{K+1} + \binom{n-1}{K}$ as

$$\binom{n-1}{K} = \binom{n}{K+1} - \binom{n-1}{K+1}\,.$$

Summing over n, and using that $\binom{0}{K+1} = 0$ [since $K+1$ is positive] shows that

1c: $$\widehat{\mathfrak{B}_K} = \mathfrak{B}_{K+1}\,.$$

The binomial polys $\{\mathfrak{B}_K\}_{K=0}^{\infty}$ form a basis for the vectorspace of polys. Since the $f \mapsto \widehat{f}$ map is linear, we can compute the summation-poly of arbitrary polynomials.

[Aside: Stronger, collection $\{\mathfrak{B}_K\}_{K=0}^{\infty}$ is a $\mathbb{Z}$-basis for the set of $\mathbb{Z}$-coeff polynomials; however, this fact isn't obvious.]

Combinatorial graphs. Some notation:

General graphs, G, H, D, S. Empty graph: Emp_N.

Complete graph: K_N. Complete bipartite graph: $K_{3,5}$.

Cyclic graph: C_N. Wheel graph: W_{N+1}, is a vertex attached to each of the N vertices of C_N.

Path-graph P, which is a special case of a tree-graph, T_N.

Named graph, e.g, *Tetrahedron*.

Comb II quizzes so far...

PB: Fri. 19Jan With $a_n := 1 + n^2$, its

OGF $A(x) := \sum_{n=0}^{\infty} a_n x^n =$ ....

[*Hint:* $x\frac{d}{dx}$-trick]

PC: Wed. 31Jan *Am I in class today?*

circle one *"Yes!"* *"Of course!"*

PD: Mon. 05Feb Perm $\alpha \in \mathbb{S}_7$ is $(7\ 1\ 4\ 2\ 6\ 3\ 5)$. A particular permutation β satisfying $\beta^3 = \alpha$, is

$\beta = (7$ .... $)$.

And Sgn(β) is (circle): $+\mathbf{1}$ $-\mathbf{1}$.

PE: Fri. 12Feb The coeff of $x^7 y^{12}$

in $[5x + y^3 + 1]^{30}$ is ....

[You may write in form number times multinomial-coeff. You can leave the multinomial-coeff as such, or write ITOf factorials.]

PF: Fri. 02Mar The # of (vertex-labeled)

forests on $[1..50]$ is ....

PG: Wed. 14Mar The number of (labeled) spanning trees

on $[1..7]$ is ....

On $[1..N]$, the number of (labeled) **rooted** spanning *forests* with $\geqslant$2 components is ....

PH: Wed. 28Mar The girls' prefs are:

$$G1\begin{bmatrix}B2\\B1\\B5\\B4\\B3\end{bmatrix},\quad G2\begin{bmatrix}B2\\B1\\B5\\B4\\B3\end{bmatrix},\quad G3\begin{bmatrix}B5\\B2\\B3\\B4\\B1\end{bmatrix},\quad G4\begin{bmatrix}B5\\B2\\B3\\B1\\B4\end{bmatrix},\quad G5\begin{bmatrix}B2\\B5\\B4\\B3\\B1\end{bmatrix}.$$

The boys' prefs are:

$$B1\begin{bmatrix}G1\\G2\\G3\\G4\\G5\end{bmatrix},\quad B2\begin{bmatrix}G4\\G5\\G3\\G2\\G1\end{bmatrix},\quad B3\begin{bmatrix}G5\\G4\\G3\\G2\\G1\end{bmatrix},\quad B4\begin{bmatrix}G3\\G5\\G1\\G4\\G2\end{bmatrix},\quad B5\begin{bmatrix}G1\\G2\\G3\\G4\\G5\end{bmatrix}.$$

Show the steps to compute a Stable-Matching, and the result, when the boys ask the girls. Ditto, when the girls ask the boys.

That's All, Folks! (Semester 2)

Comb I quizzes

P1: Wed. 27Sep Let G_n be the number of coin-flip sequences of length $2n$, with "first-return-to-zero" happening at time $2n$. Let F_n count those such sequences that start with HEADS.

Then $F_{50} =$ ________________.

P2: Mon. 02Oct The number of diagonal lattice-paths from $(0,0)$ to $(31,7)$ which *never* touch the $y{=}10$ line is ________________.

P3: Wed. 04Oct Stirling's Formula says: $n! \sim$ ________________.

Am I in class today?

circle one *"Yes!"* *"Of course!"*

P4: Mon. 09Oct The 4^{th} Bell number is 15. List, in number-of-atoms order, the 15 partitions of $\{1,2,3,4\}$.

P5: Wed. 11Oct Permutation β has cycle-signature $\lceil 5^9 \rfloor$.

It has ________________ many $\mathbf{4^{th}}$-roots with sig $\lceil 5^9 \rfloor$,

and ________________ with sig $\lceil 20^2, 5^1 \rfloor$.

[Answers can be a product of multinomials, powers, numbers.]

P6: Wed. 25Oct *The Broken Problem:* Handed-in = 30pts.

Permutation β has cycle-signature $\lceil 5^9 \rfloor$.

The # of β-$\mathbf{5^{th}}$-roots with sig $\lceil 20^1, 15^1, 5^2 \rfloor$ is: ________.

P7: Fri. 27Oct Bipartite graph $K_{7,2}$ is Eulerian. *T* *F*

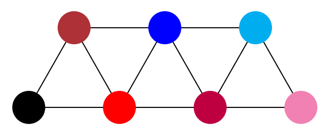
The number of permutations of $[1..6]$ which have *neither* $(4\ 1)$ *nor* $(6\ 3\ 2)$ is ________________.

[Use Incl-Excl form: $\square - \square + \square - \square + \ldots$ as appropriate.]

PBonus: Fri. 17Nov The chromatic-poly of a tree T with 5 vertices is $\mathcal{P}_T(x) =$ ________________.

In $\Omega := [1..100]$, the number of elements which are a multiple of 3 or 5 (or both) is ________________.

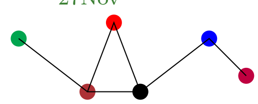
P8: Mon. 13Nov



Truss exercise: Graph has *Chromatic poly* $\mathcal{P}(x) =$ ________________.

[Can be done by inspection. Express in *chromatic form*.]

P9: Mon. 27Nov *Bird with a broken wing*: Graph



has *Chromatic poly* $\mathcal{P}(x) =$ ________________.

[Can be done by inspection. Do not bother to multiply out.]

PA: Wed. 29Nov The number of labeled 6-vertex graphs of type shown on blackboard, is:

A: ________________.

B: ________________.

C: ________________.

That's All, Folks! (Semester 1)