Primer on Cardinalities

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
28 November, 2018 (at *23:17*)

1a: *Defn: Exponentiation.* Given sets D and C, logicians define the symbol C^D as

$$C^D := \Big\{\text{The set of } \underline{\text{all}} \text{ fncs from } D \to C\Big\}.$$

These functions have **Domain** D and **Co-domain** C. Note well that C^D is a *set of functions*. For example, let $\Omega := \{\alpha, \beta, \gamma\}$ and $B := \{0, 1\}$. Then B^Ω comprises the $2^3 = 8$ many fncs

$$\begin{array}{ll} \alpha,\beta,\gamma \mapsto 0,0,0; & \alpha,\beta,\gamma \mapsto 1,0,0; \\ \alpha,\beta,\gamma \mapsto 0,0,1; & \alpha,\beta,\gamma \mapsto 1,0,1; \\ \alpha,\beta,\gamma \mapsto 0,1,0; & \alpha,\beta,\gamma \mapsto 1,1,0; \\ \alpha,\beta,\gamma \mapsto 0,1,1; & \alpha,\beta,\gamma \mapsto 1,1,1. \end{array}$$

In contrast, Ω^B comprises these $3^2 = 9$ fncs:

$$\begin{array}{lll} 0,1 \mapsto \alpha,\alpha; & 0,1 \mapsto \beta,\alpha; & 0,1 \mapsto \gamma,\alpha; \\ 0,1 \mapsto \alpha,\beta; & 0,1 \mapsto \beta,\beta; & 0,1 \mapsto \gamma,\beta; \\ 0,1 \mapsto \alpha,\gamma; & 0,1 \mapsto \beta,\gamma; & 0,1 \mapsto \gamma,\gamma. \end{array}$$

Note, for finite sets P and Q, that $|P^Q| = |P|^{|Q|}$. It is for that reason that logicians use this Set^{Set} notation. [**N.B:** Consider sets $A \asymp B$ with $A \neq B$. Although sets A^B and B^A have the same *cardinality*, they are <u>*not*</u> the same set; this, since the fncs in A^B have B as their domain, whereas those in B^A have A as their domain, yet $B \neq A$.] □

1b: *Defn: Powerset.* The ***powerset*** of a set Ω, written $\mathcal{P}(\Omega)$, is the set of <u>*all*</u> subsets of Ω. *Why do logicians sometimes write* $\{0,1\}^\Omega$ *to mean* $\mathcal{P}(\Omega)$ *?*

Well, there is a natural bijection between the two: A function $f\colon\Omega\to\{0,1\}$ yields a subset $S_f \subset \Omega$ by $S_f := \{x \in \Omega \mid f(x) = 1\}$. Easily, the map $f \mapsto S_f$ is a bijection from $\{0,1\}^\Omega$ onto $\mathcal{P}(\Omega)$.

By the way, logicians sometimes abbreviate $\{0,1\}^\Omega$ as $\mathbf{2}^\Omega$, since all that was important about the base set, $\{0,1\}$, was that it had 2 elements; it was not important what those elements were. □

1c: Lemma. *Consider sets $P \asymp \widetilde{P}$ and $Q \asymp \widetilde{Q}$. Then $P^Q \asymp \widetilde{P}^{\widetilde{Q}}$.* *Proof.* Exercise 1. ◊

ENTRANCE. Two sets A and B are ***equinumerous***, or "***bijective*** with each other", if *there exists* a bijection $A \hookrightarrow\!\!\!\!\!\twoheadrightarrow B$. [BTWay, we use a hook-arrow to indicate an injection, e.g, $A \hookrightarrow B$, and a doublehead-arrow, e.g $A \twoheadrightarrow B$ to indicate a surjection. Hence $\hookrightarrow\!\!\!\!\!\twoheadrightarrow$ indicates a bijection.] Write the equinumerous relation as

$$A \asymp B.$$

Alternatively, write $A \preccurlyeq B$ if *there exists* an injection $A \hookrightarrow B$. Finally, let $A \prec B$ mean that $A \preccurlyeq B$ yet $A \not\asymp B$.

Easily, $\asymp$ is an equiv-relation, and both $\preccurlyeq$ and $\prec$ are partial-orders on the class of cardinalities.

Call S ***countably-infinite*** or ***denumerable*** if $S \asymp \mathbb{N}$. A set S is ***countable*** if $S \preccurlyeq \mathbb{N}$, i.e, S is bijective with some subset of $\mathbb{N}$. [So a countable set is either *finite* or *countably-infinite*.] □

2: Countable-card Theorem. *Below, S represents an arbitrary <u>non-void</u> countable set.*

a: *An arbitrary subset of a countable set is countable. In particular, an arbitrary infinite subset of a countable set is countably-infinite.*

b: *Each of these is countably-infinite:* $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}\times\mathbb{N}$, $S\times\mathbb{N}$.

c: *A union of countably many countable sets is countable.* ◊

3a: *Defn.* In referring to intervals, let ***LCRO*** mean "Left-Closed Right-Open" and let ***LORC*** mean "Left-Open Right-Closed".

Use $\overline{\mathbb{R}} := [-\infty, +\infty] := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ to denote the ***extended reals***. In $\overline{\mathbb{R}}$, as an example, the set $[-\infty, 5)$ is a LCRO-interval, and $(7, \infty]$ is a LORC-interval. Both $[-4, \infty]$ and $\overline{\mathbb{R}}$ itself are closed intervals. All of these examples are *unbounded* intervals, whereas $(-4, 7]$ is a bounded interval.

In the theorem below, the word "*interval*" means "*non-trivial interval*". That is, we exclude 1-point closed intervals, e.g $[7,7] \stackrel{\text{note}}{=\!=} \{7\}$, as well as the empty interval, e.g $(7,7) \stackrel{\text{note}}{=\!=} (7,7] \stackrel{\text{note}}{=\!=} [7,7)$, each of which is the emptyset. [BTWay, the emptyset is open.] □

3b: Interval-card Theorem. *Each non-trivial sub-interval of $\overline{\mathbb{R}}$ is equinumerous with $\mathbb{R}$.* ♢

Proof. For $k = 1{,}2$, consider intervals $J_k := [a_k, b_k]$, with positive (finite) lengths $L_k := b_k - a_k$. Then the affine map

$$x \;\mapsto\; a_2 + \tfrac{L_2}{L_1}\cdot[x - a_1]$$

bijects J_1 onto J_2. The same map works for two open intervals, two LCRO-intervals, or two LORC-intervals.

Let $\mathcal{H} := \{\frac{1}{n}\}_{n=1}^{\infty}$ comprise the harmonic numbers. Define $f\colon[0,1]\to[0,1)$ as follows.[♡1]

3c:
For $x \in \mathcal{H}$: Let $n := \frac{1}{x}$, then map $x \mapsto \frac{1}{n+1}$.
For $x \notin \mathcal{H}$: Map $x \mapsto x$.

This f is a bijection. And $g(x) := 1 - x$ bijects $[0,1)$ onto $(0,1]$. Defining $h\colon(0,1]\to(0,1)$ by rule (3c) also gives a bijection. Hence

$$[0,1] \overset{f}{\hookrightarrow\!\!\!\!\rightarrow} [0,1) \overset{g}{\hookrightarrow\!\!\!\!\rightarrow} (0,1] \overset{h}{\hookrightarrow\!\!\!\!\rightarrow} (0,1)\,. \quad \textit{Thus:}$$

$\boxed{\textit{All bounded intervals have the same cardinality.}}$

Unbounded intervals. Extending $\tan()$ to $\overline{\mathbb{R}}$, note that tan bijects $J := [-\frac{\pi}{2}, \frac{\pi}{2}]$. onto the closed interval $\overline{\mathbb{R}}$. Consequently, every (bounded or unbounded) closed/open/LORO/LCRO sub-interval of $\overline{\mathbb{R}}$ is carried by arctan to a corresponding subinterval of J. Since these latter intervals are all bounded intervals, our earlier argument shows that they are all bijective with each other. ♦

4: Cantor's diagonalization thm.

i: *Firstly, $\mathbb{N} \prec \mathbb{R}$. Moreover, for each fnc $f\colon\mathbb{N}\to\mathbb{R}$ there is an explicit construction of a point $b_f \in \mathbb{R}$ which is not in* Range(f).

ii: *Every set S satisfies that $S \prec \mathcal{P}(S)$. Moreover, for each fnc $f\colon S\to\mathcal{P}(S)$, this set,*

$$\mathcal{B}ad_f \quad := \quad \{z \in S \mid f(z) \not\ni z\}$$

is not in Range(f). ♢

[♡1]This (3c) uses one of our "*Cantor's–Hotel maps*" from class.

Proof of (ii). Injection $x \mapsto \{x\}$ shows that $S \preccurlyeq \mathcal{P}(S)$.

To see that $S \not\asymp \mathcal{P}(S)$, note: *For each $z \in S$, the symmetric-difference $\mathcal{B}ad_f \triangle f(z)$ owns z; hence $\mathcal{B}ad_f$ differs from $f(z)$.* ♦

5: Card-Exponentiation Lemma (CE-Lemma). *Consider any three sets Ω, B and C. Then $\Omega^{B\times C} \asymp [\Omega^B]^C$.* ♢

Proof. Define $\Theta\colon\Omega^{B\times C}\hookrightarrow\!\!\!\!\rightarrow[\Omega^B]^C$ by

$$\Theta(f) \;:=\; \Big[c \mapsto \Big[b \mapsto f\big((b,c)\big)\Big]\Big]\,.$$

Its inverse-map $\Upsilon\colon[\Omega^B]^C\hookrightarrow\!\!\!\!\rightarrow\Omega^{B\times C}$ is

$$\Upsilon(g) \;:=\; \Big[(b,c) \mapsto \big[g(c)\big](b)\Big]\,. \qquad ♦$$

Schröder-Bernstein

Here, we examine cardinality relations between two sets, $\mathbf{A}$ and $\mathbf{\Omega}$. The below arguments do not require these sets be disjoint, but the idea is easier understand when they are. At no cost, we can arrange that $\mathbf{A}$ be disjoint from $\mathbf{\Omega}$ by replacing $\mathbf{A}$ by $\mathbf{A}\times\{1\}$, and replacing $\mathbf{\Omega}$ by $\mathbf{\Omega}\times\{2\}$.

6.1: *Consider a map $f\colon\mathbf{A}\to\mathbf{\Omega}$, an injection $g\colon\mathbf{\Omega}\hookrightarrow\mathbf{A}$, and a subset $Y \subset g(\mathbf{\Omega}) \overset{\text{note}}{\subset} \mathbf{A}$. These determine a function $\theta\colon\mathbf{A}\to\mathbf{\Omega}$ by setting $X := \mathbf{A} \smallsetminus Y$, and defining*

$$\theta|_Y := g^{-1}|_Y, \quad\text{and}\quad \theta|_X := f|_X\,.$$

Let $[\![f, g\colon Y]\!]$ denote this function θ.

Given f, g and Y as above, with f an *injection*, say that "the set Y is (f,g)-***good***" if the resulting function $\theta\colon\mathbf{A}\to\mathbf{\Omega}$ of (6.1) is a *bijection*, and call θ an "(f,g)-good bijection".

6.2: Weak Schröder-Bernstein thm. *For sets $\mathbf{A}$ and $\mathbf{\Omega}$: If $\mathbf{A} \preccurlyeq \mathbf{\Omega}$ and $\mathbf{A} \succcurlyeq \mathbf{\Omega}$, then $\mathbf{A} \asymp \mathbf{\Omega}$.* ♢

6.3: Schröder-Bernstein Thm [S-B Thm]. *Fix injections*

$$\mathbf{A}\overset{f}{\hookrightarrow}\mathbf{\Omega} \quad\textit{and}\quad \mathbf{A}\overset{g}{\hookleftarrow}\mathbf{\Omega}\,.$$

Then there exists an (f,g)-good set.

Indeed, defining the collection of points in each set with no pre-image in the other set,

$$\widehat{A} \;:=\; \mathbf{A} \smallsetminus g(\mathbf{\Omega}) \quad\text{and}\quad \widehat{\Omega} \;:=\; \mathbf{\Omega} \smallsetminus f(\mathbf{A})\,,$$

we have this: The Smallest and Largest, in the sense of inclusion, (f,g)-good sets are

6.4: $\mathcal{S}_{\langle f,g\rangle} \;:=\; \bigcup_{n=0}^{\infty} [g\circ f]^{\circ n}(g(\widehat{\Omega}))$ and

6.5: $\mathcal{L}_{\langle f,g\rangle} \;:=\; \mathbf{A} \smallsetminus \Big[\bigcup_{n=0}^{\infty} [g\circ f]^{\circ n}(\widehat{A})\Big]$,

resp.. In particular, $\mathcal{S}_{\langle f,g\rangle} \subset \mathcal{L}_{\langle f,g\rangle} \subset \mathrm{Range}(g)$. ♢

Proof. Use the 4-orbit picture from class. ♦

Bits

Define half-open and open intervals in the reals,

$$L \;:=\; [0,1) \quad\text{and}\quad R \;:=\; (0,1] \quad\text{and}\quad J \;:=\; (0,1)\,.$$

Recall that each dyadic rational has two binary numerals; the remaining reals each have one binary numeral. Define the all-zero and all-one names

$$\overline{\mathbf{0}} \;:=\; 000\cdots\,, \quad\text{and}\quad \overline{\mathbf{1}} \;:=\; 111\cdots\,,$$

Let EC be the set of eventually constant-1 or eventually constant-0 names.

Below, I'll use ***"word"*** for a *finite* string bits, e.g "10010". I'll use ***"name"*** for a (one-sided) *infinite* string, e.g 101111001001001001 00... (e.g, start with "10111" then repeat the pattern "100" forever.)

Let BITS $:= 2^{\mathbb{Z}_+}$ be the set of bit-sequences

$$\vec{\mathbf{b}} \;=\; b_1 b_2 b_3 \ldots\,, \quad \textit{with each } b_j \in \{0,1\}.$$

So BITS $\asymp \mathcal{P}$(denumerable).

7a: *Defn.* Define a fnc BinOne$:(0,1]\to$BITS that produces the binary numeral of a point. Specifically, BinOne(x) is the unique bit-sequence $\vec{\mathbf{b}}$ such that

$$\sum_{n=1}^{\infty} \frac{b_n}{2^n} \;=\; x\,,$$

and: *If x is a dyadic rational, then $\vec{\mathbf{b}}$ is eventually constant 1.* [E.g BinOne$(3/4)$ = $10111\cdots$ = $10\overline{\mathbf{1}}$.]

Define ValDTer$:$BITS$\to[0,1]$ by

$$\mathrm{ValDTer}(\vec{\mathbf{b}}) \;:=\; \sum_{n=1}^{\infty} \frac{2b_n}{3^n}\,.$$

So ValDTer$(10\overline{\mathbf{1}})$ is the number whose base-3 numeral is $.2022\cdots$, is $\frac{2}{3}+\frac{1}{9}=\frac{7}{9}$. The range of ValDTer is the famouse "middle thirds" ***Cantor set***. □

7b: S-B corollary. *It is the case that $\mathbb{R} \asymp \mathcal{P}(\mathbb{N})$.* ♢

Proof. Inject $\mathbb{R}\hookrightarrow$BITS by

$$\mathbb{R} \asymp (0,1) \xrightarrow{\mathrm{BinOne}} \mathrm{BITS}\,.$$

In the other direction,

$$\mathrm{BITS} \xrightarrow{\mathrm{ValDTer}} [0,1] \hookrightarrow \mathbb{R}\,,$$

injecting BITS$\hookrightarrow\mathbb{R}$. Now apply Schröder-Bernstein. ♦

8: Power-of-reals Thm.

i: For each posint k, we have that $\mathbb{R}^k \asymp \mathbb{R}$.

ii: Set $\mathbb{R}^{\mathbb{N}}$ is equinumerous with $\mathbb{R}$. ♢

Proof. For a cardinal $\Lambda \in \mathbb{Z}_+ \cup \{\mathbb{N}\}$, pick your favorite bijection $\mathbb{N}\times\Lambda \hookrightarrow\!\!\!\!\rightarrow \mathbb{N}$, and lift it to a bijection $f:\mathbb{R}^{\mathbb{N}\times\Lambda} \hookrightarrow\!\!\!\!\rightarrow \mathbb{R}^{\mathbb{N}}$. Finally

$$\mathbb{R}^{\Lambda} \overset{\text{by (7b)}}{\asymp} [2^{\mathbb{N}}]^{\Lambda} \overset{\text{by CE}}{\asymp} \mathbb{R}^{\mathbb{N}\times\Lambda} \overset{f}{\hookrightarrow\!\!\!\!\rightarrow} \mathbb{R}^{\mathbb{N}}\,.$$ ♦

9: Continuous-fncs Thm.

i: Firstly, $\mathbb{R}^{\mathbb{R}} \asymp 2^{\mathbb{R}} \overset{note}{\asymp} \left\{\begin{matrix}\textit{Functions only taking}\\ \textit{on values 5 and 7.}\end{matrix}\right\}$.

ii: Also, $\mathbf{C}(\mathbb{R}\to\mathbb{R}) \asymp \mathbb{R} \overset{note}{\asymp} \{\textit{Constant fncs}\}$. $\diamondsuit$

Pf of (i). $\mathbb{R}^{\mathbb{R}} \asymp [2^{\mathbb{N}}]^{\mathbb{R}} \asymp 2^{\mathbb{N}\times\mathbb{R}}$, by CE-Lem (5). Etc. $\blacklozenge$

Pf of (ii). An injection $\mathbb{R}\hookrightarrow\mathbf{C}(\mathbb{R}\to\mathbb{R})$ is $p \mapsto [x\mapsto p]$, e.g, the number 4 maps to the constant-fnc-4. Courtesy Schröder-Bernstein, then, ISTProduce an injection in the other direction. Happily,

$$\mathbf{C}(\mathbb{R}\to\mathbb{R}) \overset{\text{Lem (9a)}}{\preccurlyeq} \mathbb{R}^{\mathbb{Q}} \asymp [2^{\mathbb{N}}]^{\mathbb{N}} \overset{\text{by CE}}{\asymp} 2^{\mathbb{N}\times\mathbb{N}} \overset{\text{Etc.}}{\asymp} \mathbb{R}. \blacklozenge$$

9a: Lemma. *The mapping* $\mathbf{C}(\mathbb{R}\to\mathbb{R}) \to \mathbb{R}^{\mathbb{Q}}$, *defined by restriction* $f \mapsto f\restriction_{\mathbb{Q}}$, *is an injection.* $\diamondsuit$

Proof. Letting $h := f\restriction_{\mathbb{Q}}$, we need to recover f from h. Given a point $p\in\mathbb{R}$, take a sequence $\vec{\mathbf{q}}$ of rationals s.t $q_n \to p$. Our [unknown] f is cts at p, so

$$f(p) = \lim_{n\to\infty} f(q_n) \overset{\text{def}}{=\!=} \lim_{n\to\infty} h(q_n). \quad \blacklozenge$$

Bijection $\mathbb{R}\times\mathbb{R}$ with $\mathbb{R}$

We will show $L\times L \asymp L$. [Interweaving argument.]

Explicit bijection $\psi{:}\mathbb{R}\to\mathcal{P}(\mathbb{N})$

Let Dy be this sequence of dyadic rationals in half-open interval R,

$$\text{Dy} = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \dots \frac{15}{16}, \frac{1}{32}, \frac{3}{32}, \dots \frac{31}{32}, \frac{1}{64}, \dots\right).$$

We define a map $\mu{:}\,\text{Ec}\to\text{Dy}$, as follows.

Unfinished: as of 28Nov2018

```
/---------------------------------\
 Event-const.                Dyadic
      y           |-->        f(y)
 ==================================
      N           |-->        1/1
      Z           |-->        1/2
      0N          |-->        1/4
      1Z          |-->        3/4
      00N         |-->        1/8
      01Z         |-->        3/8
      10N         |-->        5/8
      11Z         |-->        7/8
      000N        |-->        1/16
      001Z        |-->        3/16
      010N        |-->        5/16
      011Z        |-->        7/16
      100N        |-->        9/16
      101Z        |-->       11/16
      110N        |-->       13/16
      111Z        |-->       15/16
      0000N       |-->        1/32
      0001Z       |-->        3/32
      0010N       |-->        5/32
      0011Z       |-->        7/32
      ...                     ...
      1111Z       |-->       31/32
      00000N      |-->        1/64
      00001N      |-->        3/64
      00010N      |-->        5/64
      ...                     ...
      11110Z      |-->       61/64
      11111Z      |-->       63/64
      000000N     |-->        1/128
      000001N     |-->        3/128
      ...                     ...
             Et cetera
\_________________________________/
```

Algebraic numbers

(Defn of the set $\mathbb{A}$ of algebraic numbers. The set $\mathbb{R} \smallsetminus \mathbb{A}$ of real transcendentals, and $\mathbb{C} \smallsetminus \mathbb{A}$ of transcendental in general. Examples of transcendental numbers. Possible discussion of Liouville's thm)

10: Algebraic-numbers Thm. *The set $\mathbb{A}$ of algebraic numbers is denumerable.* ♢

Proof. (The set of intpolys is countable, Etc.) ♦

Filename: `Problems/SetTheory/primer-cardinality.latex`
As of: *Tuesday 08Apr2014.* Typeset: *28Nov2018* at *23:17.*