

Primer on Cardinalities

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1a: Defn: Exponentiation. Given sets D and C , logicians define the symbol C^D as

$$C^D := \left\{ \text{The set of all fncs from } D \rightarrow C \right\}.$$

These functions have **Domain** D and **Co-domain** C . Note well that C^D is a *set of functions*. For example, let $\Omega := \{\alpha, \beta, \gamma\}$ and $B := \{0, 1\}$. Then B^Ω comprises the $2^3 = 8$ many fncs

$$\begin{array}{ll} \alpha, \beta, \gamma \mapsto 0, 0, 0; & \alpha, \beta, \gamma \mapsto 1, 0, 0; \\ \alpha, \beta, \gamma \mapsto 0, 0, 1; & \alpha, \beta, \gamma \mapsto 1, 0, 1; \\ \alpha, \beta, \gamma \mapsto 0, 1, 0; & \alpha, \beta, \gamma \mapsto 1, 1, 0; \\ \alpha, \beta, \gamma \mapsto 0, 1, 1; & \alpha, \beta, \gamma \mapsto 1, 1, 1. \end{array}$$

In contrast, Ω^B comprises these $3^2 = 9$ fncs:

$$\begin{array}{lll} 0, 1 \mapsto \alpha, \alpha; & 0, 1 \mapsto \beta, \alpha; & 0, 1 \mapsto \gamma, \alpha; \\ 0, 1 \mapsto \alpha, \beta; & 0, 1 \mapsto \beta, \beta; & 0, 1 \mapsto \gamma, \beta; \\ 0, 1 \mapsto \alpha, \gamma; & 0, 1 \mapsto \beta, \gamma; & 0, 1 \mapsto \gamma, \gamma. \end{array}$$

Note, for finite sets P and Q , that $|P^Q| = |P|^{|Q|}$. It is for that reason that logicians use this Set^{Set} notation. **[N.B:** Consider sets $A \asymp B$ with $A \neq B$. Although sets A^B and B^A have the same *cardinality*, they are *not* the same set; this, since the fncs in A^B have B as their domain, whereas those in B^A have A as their domain, yet $B \neq A$.] \square

1b: Defn: Powerset. The **powerset** of a set Ω , written $\mathcal{P}(\Omega)$, is the set of *all* subsets of Ω . Why do logicians sometimes write $\{0, 1\}^\Omega$ to mean $\mathcal{P}(\Omega)$?

Well, there is a natural bijection between the two: A function $f: \Omega \rightarrow \{0, 1\}$ yields a subset $S_f \subset \Omega$ by $S_f := \{x \in \Omega \mid f(x) = 1\}$. Easily, the map $f \mapsto S_f$ is a bijection from $\{0, 1\}^\Omega$ onto $\mathcal{P}(\Omega)$.

By the way, logicians sometimes abbreviate $\{0, 1\}^\Omega$ as 2^Ω , since all that was important about the base set, $\{0, 1\}$, was that it had 2 elements; it was not important what those elements were. \square

1c: Lemma. Consider sets $P \asymp \tilde{P}$ and $Q \asymp \tilde{Q}$. Then $P^Q \asymp \tilde{P}^{\tilde{Q}}$. **Proof.** Exercise 1. \diamond

ENTRANCE. Two sets A and B are **equinumerous**, or “**bijective** with each other”, if *there exists* a bijection $A \leftrightarrow B$. [BTWay, we use a hook-arrow to indicate an injection, e.g. $A \hookrightarrow B$, and a doublehead-arrow, e.g. $A \leftrightarrow B$ to indicate a surjection. Hence \leftrightarrow indicates a bijection.] Write the **equinumerous** relation as

$$A \asymp B.$$

Alternatively, write $A \preceq B$ if *there exists* an **injection** $A \hookrightarrow B$. Finally, let $A \prec B$ mean that $A \preceq B$ yet $A \not\asymp B$.

Easily, \asymp is an equiv-relation, and both \preceq and \prec are partial-orders on the class of cardinalities.

Call S **countably-infinite** or **denumerable** if $S \asymp \mathbb{N}$. A set S is **countable** if $S \preceq \mathbb{N}$, i.e. S is bijective with some subset of \mathbb{N} . [So a countable set is either *finite* or *countably-infinite*.] \square

2: Countable-card Theorem. Below, S represents an arbitrary non-void countable set.

a: An arbitrary subset of a countable set is countable. In particular, an arbitrary infinite subset of a countable set is countably-infinite.

b: Each of these is countably-infinite: $\mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}, S \times \mathbb{N}$.

c: A union of countably many countable sets is countable. \diamond

3a: Defn. In referring to intervals, let **LCRO** mean “Left-Closed Right-Open” and let **LORC** mean “Left-Open Right-Closed”.

Use $\overline{\mathbb{R}} := [-\infty, +\infty] := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ to denote the **extended reals**. In $\overline{\mathbb{R}}$, as an example, the set $[-\infty, 5)$ is a LCRO-interval, and $(7, \infty]$ is a LORC-interval. Both $[-4, \infty]$ and $\overline{\mathbb{R}}$ itself are closed intervals. All of these examples are *unbounded* intervals, whereas $(-4, 7]$ is a bounded interval.

In the theorem below, the word “*interval*” means “*non-trivial interval*”. That is, we exclude 1-point closed intervals, e.g. $[7, 7] \stackrel{\text{note}}{=} \{7\}$, as well as the empty interval, e.g. $(7, 7) \stackrel{\text{note}}{=} (7, 7) \stackrel{\text{note}}{=} [7, 7)$, each of which is the emptyset. [BTWay, the emptyset is open.] \square

3b: Interval-card Theorem. Each non-trivial sub-interval of $\overline{\mathbb{R}}$ is equinumerous with \mathbb{R} . \diamond

Proof. For $k = 1, 2$, consider intervals $J_k := [a_k, b_k]$, with finite lengths $L_k := b_k - a_k$. Then the affine map

$$x \mapsto a_2 + \frac{L_2}{L_1} \cdot [x - a_1]$$

bijects J_1 onto J_2 . The same map works for two open intervals, two LCRO-intervals, or two LORC-intervals.

Let $\mathcal{H} := \{\frac{1}{n}\}_{n=1}^\infty$ comprise the harmonic numbers. Define $f: [0, 1] \rightarrow [0, 1)$ as follows. \heartsuit^1

3c: For $x \in \mathcal{H}$: Let $n := \frac{1}{x}$, then map $x \mapsto \frac{1}{n+1}$.
 For $x \notin \mathcal{H}$: Map $x \mapsto x$.

This f is a bijection. And $g(x) := 1 - x$ bijects $[0, 1)$ onto $(0, 1]$. Defining $h: (0, 1] \rightarrow (0, 1)$ by rule (3c) also gives a bijection. Hence

$$[0, 1] \xrightarrow{f} [0, 1) \xrightarrow{g} (0, 1] \xrightarrow{h} (0, 1). \quad \text{Thus:}$$

All bounded intervals have the same cardinality.

Unbounded intervals. Extending $\tan()$ to $\overline{\mathbb{R}}$, note that \tan bijects $J := [-\frac{\pi}{2}, \frac{\pi}{2}]$. onto the closed interval $\overline{\mathbb{R}}$. Consequently, every (bounded or unbounded) closed/open/LORO/LCRO sub-interval of $\overline{\mathbb{R}}$ is carried by \arctan to a corresponding subinterval of J . Since these latter intervals are all bounded intervals, our earlier argument shows that they are all bijective with each other. \diamond

4: Cantor's diagonalization thm.

i: Firstly, $\mathbb{N} \prec \mathbb{R}$. Moreover, for each fnc $f: \mathbb{N} \rightarrow \mathbb{R}$ there is an explicit construction of a point $b_f \in \mathbb{R}$ which is not in $\text{Range}(f)$.

ii: Every set S satisfies that $S \prec \mathcal{P}(S)$. Moreover, for each fnc $f: S \rightarrow \mathcal{P}(S)$, this set,

$$\mathcal{B}ad_f := \{z \in S \mid f(z) \ni z\}$$

is not in $\text{Range}(f)$. \diamond

\heartsuit^1 This (3c) uses one of our “Cantor’s–Hotel maps” from class.

Proof of (ii). Injection $x \mapsto \{x\}$ shows that $S \preccurlyeq \mathcal{P}(S)$.

To see that $S \not\prec \mathcal{P}(S)$, note: For each $z \in S$, the symmetric-difference $\mathcal{B}ad_f \Delta f(z)$ owns z ; hence $\mathcal{B}ad_f$ differs from $f(z)$. \diamond

5: Card-Exponentiation Lemma (CE-Lemma). Consider any three sets Ω , B and C . Then $\Omega^{B \times C} \asymp [\Omega^B]^C$.

Proof. Exercise 2.

Schröder-Bernstein

Here, we examine cardinality relations between two sets, \mathbf{A} and Ω . The below arguments do not require these sets be disjoint, but the idea is easier understand when they are. At no cost, we can arrange that \mathbf{A} be disjoint from Ω by replacing \mathbf{A} by $\mathbf{A} \times \{1\}$, and replacing Ω by $\Omega \times \{2\}$.

Consider a map $f: \mathbf{A} \rightarrow \Omega$, an injection $g: \Omega \rightarrow \mathbf{A}$, and a subset $Y \subset g(\Omega) \stackrel{\text{note}}{\subset} \mathbf{A}$.

6.1: These determine a function $\theta: \mathbf{A} \rightarrow \Omega$ by setting $X := \mathbf{A} \setminus Y$, and defining

$$\theta \downarrow_Y := g^{-1} \downarrow_Y, \quad \text{and} \quad \theta \downarrow_X := f \downarrow_X.$$

Let $\llbracket f, g: Y \rrbracket$ denote this function θ .

Given f, g and Y as above, with f an injection, say that “the set Y is **(f, g)-good**” if the resulting function $\theta: \mathbf{A} \rightarrow \Omega$ of (6.1) is a bijection, and call θ an “**(f, g)-good bijection**”.

6.2: Weak Schröder-Bernstein thm. For sets \mathbf{A} and Ω : If $\mathbf{A} \preccurlyeq \Omega$ and $\mathbf{A} \succcurlyeq \Omega$, then $\mathbf{A} \asymp \Omega$. \diamond

6.3: Schröder-Bernstein Thm [S-B Thm]. Fix injections

$$\mathbf{A} \xrightarrow{f} \Omega \quad \text{and} \quad \mathbf{A} \xleftarrow{g} \Omega.$$

Then there exists an **(f, g)-good set**.

Indeed, defining the collection of points in each set with no pre-image in the other set,

$$\widehat{A} := \mathbf{A} \setminus g(\Omega) \quad \text{and} \quad \widehat{\Omega} := \Omega \setminus f(\mathbf{A}),$$

we have this: The Smallest and Largest, in the sense of inclusion, **(f, g)-good sets** are

$$6.4: \quad \mathcal{S}_{\langle f, g \rangle} := \bigcup_{n=0}^\infty [g \circ f]^{on} (g(\widehat{\Omega})) \quad \text{and}$$

$$6.5: \quad \mathcal{L}_{\langle f, g \rangle} := \mathbf{A} \setminus \left[\bigcup_{n=0}^\infty [g \circ f]^{on} (\widehat{A}) \right],$$

resp.. In particular, $\mathcal{S}_{\langle f, g \rangle} \subset \mathcal{L}_{\langle f, g \rangle} \subset \text{Range}(g)$. \diamond

Proof. Use the 4-orbit picture from class.

◆ *Proof.* Inject $\mathbb{R} \hookrightarrow \text{BITS}$ by

$$\mathbb{R} \simeq (0, 1) \xrightarrow{\text{BinOne}} \text{BITS}.$$

In the other direction,

$$\text{BITS} \xrightarrow{\text{ValDTer}} [0, 1] \hookrightarrow \mathbb{R},$$

injecting $\text{BITS} \hookrightarrow \mathbb{R}$. Now apply Schröder-Bernstein. ◆

Bits

Define half-open and open intervals in the reals,

$$L := [0, 1) \quad \text{and} \quad R := (0, 1] \quad \text{and} \quad J := (0, 1).$$

Recall that each dyadic rational has two binary numerals; the remaining reals each have one binary numeral. Define the all-zero and all-one names

$$\bar{0} := 000\dots, \quad \text{and} \quad \bar{1} := 111\dots,$$

Let EC be the set of eventually constant-1 or eventually constant-0 names.

Below, I'll use "**word**" for a *finite* string bits, e.g. "10010". I'll use "**name**" for a (one-sided) *infinite* string, e.g. 10111100100100100100... (e.g. start with "10111" then repeat the pattern "100" forever.)

Let $\text{BITS} := 2^{\mathbb{Z}^+}$ be the set of bit-sequences

$$\vec{b} = b_1 b_2 b_3 \dots, \quad \text{with each } b_j \in \{0, 1\}.$$

So $\text{BITS} \simeq \mathcal{P}(\text{denumerable})$.

7a: Defn. Define a fnc $\text{BinOne} : (0, 1] \rightarrow \text{BITS}$ that produces the binary numeral of a point. Specifically, $\text{BinOne}(x)$ is the unique bit-sequence \vec{b} such that

$$\sum_{n=1}^{\infty} \frac{b_n}{2^n} = x,$$

and: *If x is a dyadic rational, then \vec{b} is eventually constant 1.* [E.g. $\text{BinOne}(3/4) = 10111\dots = 10\bar{1}$.]

Define $\text{ValDTer} : \text{BITS} \rightarrow [0, 1]$ by

$$\text{ValDTer}(\vec{b}) := \sum_{n=1}^{\infty} \frac{2b_n}{3^n}.$$

So $\text{ValDTer}(10\bar{1})$ is the number whose base-3 numeral is $.2022\dots$, is $\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$. The range of ValDTer is the famous "middle thirds" **Cantor set**. □

8: Power-of-reals Thm.

i: For each posint k , we have that $\mathbb{R}^k \simeq \mathbb{R}$.

ii: Set $\mathbb{R}^{\mathbb{N}}$ is equinumerous with \mathbb{R} . ◆

Proof. For a cardinal $\Lambda \in \mathbb{Z}_+ \cup \{\mathbb{N}\}$, pick your favorite bijection $\mathbb{N} \times \Lambda \hookrightarrow \mathbb{N}$, and lift it to a bijection $f : \mathbb{R}^{\mathbb{N} \times \Lambda} \hookrightarrow \mathbb{R}^{\mathbb{N}}$. Finally

$$\mathbb{R}^{\Lambda} \underset{\text{by (7b)}}{\simeq} [2^{\mathbb{N}}]^{\Lambda} \underset{\text{by CE}}{\simeq} \mathbb{R}^{\mathbb{N} \times \Lambda} \xrightarrow{f} \mathbb{R}^{\mathbb{N}}. \quad \blacklozenge$$

7b: S-B corollary. *It is the case that $\mathbb{R} \simeq \mathcal{P}(\mathbb{N})$.* ◆

9: Continuous-fncs Thm.

a: Firstly, $\mathbb{R}^{\mathbb{R}} \simeq 2^{\mathbb{R}} \stackrel{\text{note}}{\simeq} \left\{ \begin{array}{l} \text{Functions only taking} \\ \text{on values 0 and 1.} \end{array} \right\}$.

b: Also, $\mathbf{C}(\mathbb{R} \rightarrow \mathbb{R}) \simeq \mathbb{R} \stackrel{\text{note}}{\simeq} \{ \text{Constant fncs} \}$. \diamond

Pf of (a). $\mathbb{R}^{\mathbb{R}} \simeq [2^{\mathbb{N}}]^{\mathbb{R}} \simeq 2^{\mathbb{N} \times \mathbb{R}}$, by CE-Lem (5). Etc. \blacklozenge

Pf of (b). An injection $\mathbb{R} \hookrightarrow \mathbf{C}(\mathbb{R} \rightarrow \mathbb{R})$ is $p \mapsto [x \mapsto p]$, e.g, the number 5 maps to the constant-fnc-5. Courtesy Schröder-Bernstein, then, ISTProduce an injection in the other direction. Happily,

$$\mathbf{C}(\mathbb{R} \rightarrow \mathbb{R}) \stackrel{\text{Lem (9a)}}{\simeq} \mathbb{R}^{\mathbb{Q}} \simeq [2^{\mathbb{N}}]^{\mathbb{N}} \stackrel{\text{by CE}}{\simeq} 2^{\mathbb{N} \times \mathbb{N}} \stackrel{\text{Etc.}}{\simeq} \mathbb{R}. \blacklozenge$$

9a: Lemma. The mapping $\mathbf{C}(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}^{\mathbb{Q}}$, defined by restriction $f \mapsto f|_{\mathbb{Q}}$, is an injection. \diamond

Proof. Letting $h := f|_{\mathbb{Q}}$, we need to recover f from h . Given a point $p \in \mathbb{R}$, take a sequence \vec{q} of rationals s.t $q_n \rightarrow p$. Our [unknown] f is cts at p , so

$$f(p) = \lim_{n \rightarrow \infty} f(q_n) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} h(q_n). \blacklozenge$$

Bijection $\mathbb{R} \times \mathbb{R}$ with \mathbb{R}

We will show $L \times L \simeq L$. [Interweaving argument.]

Explicit bijection $\psi: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{N})$

Let DY be this sequence of dyadic rationals in half-open interval R ,

$$\text{DY} = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \dots, \frac{15}{16}, \frac{1}{32}, \frac{3}{32}, \dots, \frac{31}{32}, \frac{1}{64}, \dots \right).$$

We define a map $\mu: \text{EC} \rightarrow \text{DY}$, as follows.

Unfinished: as of 17Nov2017

/-----\		
Event-const.		Dyadic
y	-->	f(y)
=====		
N	-->	1/1
Z	-->	1/2

0N	-->	1/4
1Z	-->	3/4
00N	-->	1/8
01Z	-->	3/8
10N	-->	5/8
11Z	-->	7/8
000N	-->	1/16
001Z	-->	3/16
010N	-->	5/16
011Z	-->	7/16
100N	-->	9/16
101Z	-->	11/16
110N	-->	13/16
111Z	-->	15/16
0000N	-->	1/32
0001Z	-->	3/32
0010N	-->	5/32
0011Z	-->	7/32
...		...
1111Z	-->	31/32
00000N	-->	1/64
00001N	-->	3/64
00010N	-->	5/64
...		...
11110Z	-->	61/64
11111Z	-->	63/64
000000N	-->	1/128
000001N	-->	3/128
...		...

Et cetera

Algebraic numbers

(Defn of the set \mathbb{A} of algebraic numbers. The set $\mathbb{R} \setminus \mathbb{A}$ of real transcendentals, and $\mathbb{C} \setminus \mathbb{A}$ of transcendental in general. Examples of transcendental numbers. Possible discussion of Liouville's thm)

10: Algebraic-numbers Thm. The set \mathbb{A} of algebraic numbers is denumerable. \diamond

Proof. (The set of intpolys is countable, Etc.) \blacklozenge

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