An eventually derivative-zero f must be a polynomial

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Abstract: At each point x, *some* derivative of f is zero. Then f is a polynomial.

Conventions. Given a $\mathbf{C}^\infty$-fnc f on $\mathbb{R}$, let $\boldsymbol{\kappa}_f(x)$ be the infimum of natnums k st. $f^{(k)}(x)$ is zero.[♡1] Let ***interval*** henceforth mean a *non-void* open interval in X.

1: Theorem. *X is an open interval and $f{:}X\to\mathbb{R}$ is $\mathbf{C}^\infty$ such that $\boldsymbol{\kappa}=\boldsymbol{\kappa}_f$ is everywhere finite (but not necessarily bounded). Then f is a polynomial.* ◊

The Setup. Each interval has infinitely-many points. So

2: *If two polynomials agree as-functions on an interval, then they are the same polynomial.*

An interval J is ***good*** if $f\lfloor_J$ is a poly; i.e, for some natnum N the restriction $f^{(N)}\lfloor_J$ is identically-zero. Courtesy (2), the union of touching good-intervals is good. So each good interval lies inside a *maximal* good J, and each two maximal-goods are disjoint. We redefine ***good*** to mean maximally-good. Letting $\mathcal{G}$ be the collection of good intervals, our goal is to show that

$$B := X \smallsetminus \bigsqcup(\mathcal{G}), \quad \text{the set of } \textbf{\textit{bad}} \text{ points},$$

is empty. By maximality:

3: *Each endpoint of a good J is either bad or is an endpoint of X.*

(If an endpt of J isn't in X, then it must be an endpoint of X.)

Proof that $B=\varnothing$. FTSOC, suppose that $B\neq\varnothing$. Since B is closed, it is a complete metric space and so the BCT (Baire Category Thm) applies. Now

$$K_n := \{b\in B \mid f^{(n)}(b)=0\}$$

is B-closed, since $f^{(n)}$ is cts. But $\bigcup_0^\infty K_n = B$. By BCThm *some* K –say K_7– has B-interior. I.e, there is an X-interval I so that: $I\cap B$ is non-void and $f^{(7)}\equiv 0$ on $I\cap B$. We can thus redefine X to be this I and have $f^{(7)}$ is identically-zero on B.

At a good point z, note that $f^{(k)}(z)=0$ for all large k. Thus we may redefine f to be $f^{(7)}$ and now have

†: *$f()$ is identically-zero on B.*

For $\boldsymbol{\kappa}_f$ is certainly bounded at good points, and at bad points $\boldsymbol{\kappa}_f$ is zero.

Accumulation points. Could a bad point b be an isolated point of B? But then for some points $a<b<c$ our f is, say, a deg-5 poly on interval (a,b) and is a deg-8 poly on (b,c). Then $f^{(9)}$ would be identically-zero on (a,c), except at b (since b is bad). But f is continuous.

So each bad point b is an accumulation pt of B. And for bad pts $z_j\neq b$ with $z_j\to b$, the difference-quotients $\frac{f(z_j)-f(b)}{z_j-b}$ are all zero. So $f'(b)$ must be zero. Since this holds at every bad point, iterating tells us:

‡: *Each derivative of f is identically-zero on B.*

Using endpoints. Our $B\neq\varnothing$ so $f\not\equiv 0$ on X. Courtesy (‡), then, there is a good interval J on which f is not zip; say $f\lfloor_J$ has degree 5. But then $f^{(5)}$ is a non-zero constant on J. And one endpoint, b, of J must be in X; so b is bad. Thus $f^{(5)}(b)=0$, so $f^{(5)}$ is not cts at b. ♦

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[♡1] Everywhere, function $f(x) := |x^3|$ has *some* derivative equaling zero. Yet f is not a poly on $\mathbb{R}$. But also note that f'' is *not* differentiable at $x{=}0$.