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Complex Analysis Ph.D Exam

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Notes. Please write up solutions to the eight problems $C1$–$C8$, below. Please write LARGE, as the grader's eyes are older and weaker than your eyes...

A ***square*** $S \subset \mathbb{C}$ is a set of the form $[a, a+L] \times [b, b+L]$, where L is positive. Use "$s := \text{Foo}$" to mean that Foo is the *definition* of the new symbol s.

$C$1: Find complex numbers a, b, c, d, with $ad - bc \neq 0$, so that the Möbius transformation

$$\mu(z) := \frac{az+b}{cz+d}$$

carries the imaginary axis to the circle whose radius is 2 and whose center is $3 = 3 + 0\boldsymbol{i}$.

$C$2: With B the open unit ball $|z| < 1$, consider a non-constant analytic function $h: B \to \mathbb{C}$.

i Suppose that $\text{Re}(h(z)) \geqslant 0$ for each $z \in B$. Prove that the inequality can then be strengthened to "$>$".

ii With $\text{Re}(h(z)) > 0$ on B, suppose further that $h(0) = 1$. Prove, for each $z \in B$, that

$$\frac{1-|z|}{1+|z|} \leqslant |h(z)| \leqslant \frac{1+|z|}{1-|z|}.$$

Soln-$C$2: Let H be the open half-plane $\text{Re}(z) > 0$. I will show that $\forall z \in B$:

$$\textbf{1:} \qquad \frac{1-|z|}{1+|z|} \leqslant |h(z)|.$$

Since $w \mapsto \frac{1}{w}$ is an analytic map $H \to H$, it follows that $\frac{1}{h}$ maps $B \to H$. Applying (**??**) to $\frac{1}{h}$, then taking reciprocals, yields

$$\frac{1+|z|}{1-|z|} \geqslant |h(z)|,$$

which is the other requested inequality.

Proving (??) with the Schwarz Lemma. The Möbius map $\mu(w) := \frac{1-w}{1+w}$ carries H to B, sending $1 \mapsto 0$. Thus

$$f := \mu \circ h$$

is an analytic map $B \to B$ with $f(0) = 0$, and so we may apply the Schwarz lemma to conclude that $|z| \geqslant |f(z)|$, having fixed a particular point $z \in B$. That is,

$$*: \qquad |z| \geqslant \left|\frac{1-h(z)}{1+h(z)}\right|.$$

But if $M \neq 1$ is a complex number then $\frac{|1-M|}{|1+M|} \geqslant \frac{1-|M|}{1+|M|}$. Letting $W := |h(z)|$ and $Z := |z|$, then $(*)$ implies that

$$\textbf{2:} \qquad Z \geqslant \frac{1-W}{1+W} \overset{\text{note}}{=\!=} \mu(W).$$

μ reverses order on $\mathbb{R}_+$. Note that

$$\mu(w) = -1 + \frac{2}{1+w}.$$

Thus $\mu()$ is order-reversing on the positive reals. Hence (**??**) implies that

$$\textbf{??'}: \qquad \mu(Z) \leqslant \mu(\mu(W)).$$

Since μ is its own inverse function (μ is an involution), the RHS equals W. Hence (**??'**) *is* (**??**).

$C$3: ① Suppose that $P()$ is a monic polynomial with degree $N \geqslant 1$. With $\alpha_1, \ldots, \alpha_N$ an enumeration [with multiplicity] of the zeros of $P()$, suppose that $\forall k : \text{Re}(\alpha_k) > 0$.

Prove that all the zeros of the *derivative*, P', also lie in the positive half-plane, as follows: Establish that

$$\frac{P'(z)}{P(z)} = \frac{1}{z-\alpha_1} + \frac{1}{z-\alpha_2} + \cdots + \frac{1}{z-\alpha_N},$$

then use it to complete the proof.

② Prove Lucas's theorem: *If all the zeros of a non-constant polynomial P lie in a convex polygon $Q \subset \mathbb{C}$, then all the zeros of P' lie in Q.*

③ Show that (②) can *fail* if $P()$ is allowed to be a rational function: Namely, by letting

$$P(z) := \frac{z}{z^2+1},$$

find a half-plane H which owns a zero of P' but has no zero of P.

$C$4: Use the Residue Calculus to compute

$$I := \int_0^{+\infty} \frac{1}{[x^4+4] \cdot [x^2+9]^9} \, dx.$$

To save arithmetic, you may define some **explicit** points $P_1, \ldots, P_L \in \mathbb{C}$ (what should L be?) and **explicit** functions $h_1, \ldots, h_L$, and then may express your answer explicitly in the form

$$I = \Big[h_1(P_1) + \cdots + h_L(P_L)\Big] \cdot \textit{Constant}\,.$$

(Do not bother to perform the function-evaluation.)

$C5$: (a) State (but do not prove) Morera's Theorem. (You may use this without proof in (b), if you so wish.)

(b) Prove this version of the Schwarz Reflection Principle: *Suppose f is continuous in the closed upper half-plane $H := \mathbb{R} \times [0, \infty)$ and is analytic on the interior of H. Further suppose that f is real-valued on the real-axis. By defining $\Phi := f$ on H, and*

$$\Phi(z) := \overline{f(\overline{z})}, \quad \text{for all } z \in \mathbb{C} \smallsetminus H,$$

*extend f to all of $\mathbb{C}$. **Then** this Φ is analytic.*

$C6$: ❶ Please state Picard's Theorem.

❷ Let h be meromorphic in the whole complex plane. Suppose that the range of h omits three distinct values (one of them can be ∞). Prove that h is constant.

❸ Suppose that f and g are entire functions such that, on $\mathbb{C}$,

$$f^3 + g^3 = 1\,.$$

Prove that f and g are each constant functions. [*Note:* Symbol f^3 means $f \cdot f \cdot f$.]

Soln-$C7$: FTSOContradiction, suppose that g is not constant.

Letting A, B, C denote the three cube-roots of -1, note that the two-variable polynomial $x^3 + y^3$ factors as

$$[x - Ay][x - By][x - Cy]\,.$$

(To see this, view y as a constant and factor the resulting cubic of x.) Consequently, we may write

$$1 = [f - Ag][f - Bg][f - Cg]\,.$$

For each $z \in \mathbb{C}$, then,

$$*: \qquad f(z) - Ag(z) \neq 0$$

In particular, f and g have no common zeros. Thus the meromorphic function

$$h := \frac{f}{g}$$

takes the value ∞ at each zero of g. And if z is *not* a zero of g then $h(z) \neq A$, courtesy of $(*)$.

The upshot is that $\text{Range}(h)$ omits the value A. Similarly it omits (distinct) values B and C. So by the preceding part, h must be constant.

Last step. Calling this constant κ, we conclude that

$$1 = M \cdot g^3\,, \quad \text{where } M := \kappa^3 + 1.$$

Since g^3 is a non-zero constant, $\text{Range}(g)$ lies inside a 3-point set. Since $\text{Range}(g)$ is connected, g must be constant.

$C7$: Fix a real $b > 0$. Write down an entire function, f, that vanishes *precisely* on the sequence $(z_n)_{n=1}^{\infty}$, where $z_n := n^b$.

$C8$: Show that all roots of polynomial $P(z) := z^5 + 15z + 1$ lie in the ball $|z| < 2$, but that only one root satisfies $|z| < \frac{3}{2}$.

Soln-$C8$: Let C_r denote the radius-r circle, centered at the origin. Let $\text{Zeros}_r(f)$ denote the number of zeros of $f()$ enclosed by C_r.

Radius 2. Let $f(z) := z^5 + 1$. If $|z| \geqslant 2$ then $|f(z)|$ dominates $2^5 - 1 = 31$. And for each $z \in C_2$,

$$\big|P(z) - f(z)\big| = |15z| = 30 < 31\,.$$

Hence we may apply Rouche's thm to conclude that

$$\text{Zeros}_r(P) = \text{Zeros}_r(f) \stackrel{\text{note}}{=\!=} 5.$$

Radius 3/2. Now let $g(z) := 15z$. For $z \in C_{3/2}$ we note that $|15z| = \frac{45}{2} > 22$. And

$$\big|P(z) - g(z)\big| \leqslant |z^5| + 1 = \tfrac{3^5}{2^5} + 1\,.$$

We will conclude that $\text{Zeros}_r(P) = \text{Zeros}_r(g) = 1$ if Rouche's thm applies. Rouche's thm certainly will apply, if

$$\frac{3^5}{2^5} \stackrel{?}{\leqslant} 22 - 1 \stackrel{\text{note}}{=\!=} 7 \cdot 3\,.$$

I.e, if $3^4 \leqslant 7 \cdot 2^5$, which holds trivially.

End of Complex Analysis Ph.D Exam

C10: Prove the Cauchy-Goursat theorem: *Suppose that f is analytic on a square S. Then*

3: $$\int_{\partial S} f(z)\,\mathrm{d}z \;=\; 0\,,$$

where ∂S is the boundary of S oriented in the positive (counterclockwise) *direction.*

Note: You may use –without proof– that (**??**) holds when f is a polynomial. [*Suggestion:* For the sake of contradiction, suppose that there is an $\varepsilon > 0$ for which $\left|\int_{\partial S} f(z)\,\mathrm{d}z\right| > \varepsilon \cdot \text{Area}(S)$. Now subdivide S into four squares (etc.) and eventually argue that there must be a point $P \in S$ where f is not differentiable.]

C11: a State the Great Picard Theorem.

b Suppose that p and q are not-constant polynomials. Prove that the equation

$$\mathrm{e}^{p(z)} + q(z) \;=\; 0$$

has infinitely many solutions. [*JK:* We probably do not want TWO Picard's problems. Is there a standard theorem for them to cite to show that $\exp(p(z))/q(z)$] has an essential singularity at ∞?

C12: With D the disk $|z| \leqslant 1$, suppose that $u()$ is continuous on D and subharmonic on the interior of D. If

$$\frac{1}{2\pi}\int_{\partial D} u(z)\,\mathrm{d}z \;=\; u(0)\,,$$

then u is harmonic.

C13: Suppose that $u()$ is a *non-negative* harmonic function on $\mathbb{R} \times \mathbb{R}$. Prove that u is a constant function.

C14: Let S be the square with the four corner-points $(\pm 2, \pm 2)$, and let ω be a complex number properly *inside* the square.

① Locate all poles of $h(z) := \dfrac{\tan(z/2)}{[z-\omega]^6}$. At each pole which is *inside of* S, please compute the residue of h, expressing the answer as a function of ω.

② Please compute $\int_{\partial S} h(z)\,\mathrm{d}z$. (Do not bother to multiply-out factorials.)

C8: Let $f{:}\Omega{\to}\Omega_0$ be an analytic map between open subsets of $\mathbb{C}$. Suppose that $u{:}\Omega_0{\to}\mathbb{R}$ is subharmonic.

If u has continuous second-order partial derivatives, prove that

$$\Delta(u \circ f) \;=\; |f'|^2 \cdot \Delta(u) \circ f\,.$$

(The Laplacian of u, written $\Delta(u)$, means $u_{xx} + u_{yy}$.)

C6': ❶ Please state Picard's Theorem.

❷ Let f be meromorphic in the whole complex plane. Suppose that the range of f omits three distinct values (one of them can be ∞). Prove that f is constant.

❸ Suppose that f, g, h are entire functions such that

$$f^3 + g^3 \;=\; h^3\,.$$

Prove then that f and g are each of the form *Constant* $\cdot\, h$. [*Note:* Symbol f^3 means $f \cdot f \cdot f$.]