

There is one order-complete ordered-field

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Rings. In a ring $(\Gamma, +, \mathbf{0}_\Gamma, \cdot, \mathbf{1}_\Gamma)$, if there is a posint n so that $\mathbf{1}_\Gamma + \mathbf{1}_\Gamma + \dots + \mathbf{1}_\Gamma$ equals $\mathbf{0}_\Gamma$, then the *smallest* such n is called the **characteristic** of the ring, and I write $\text{Char}(\Gamma) = n$. If no such posint exists, then I will write $\text{Char}(\Gamma) = \infty$; however, the *standard term* is $\text{Char}(\Gamma) = 0$, and you will see this in algebra texts and in some of my notes.

A ring is **commutative** (abbrev., **comm-ring**) if its multiplication is commutative. In a comm-ring Γ , a **zero-divisor** $\alpha \in \Gamma$ admits a non-zero elt $\beta \in \Gamma$ (this β need not be unique) so that $\alpha\beta = \mathbf{0}_\Gamma$. Use **ZD** for **zero-divisor**. [Letting \equiv denote \equiv_{12} , in the \mathbb{Z}_{12} ring, 9 is a ZD, since $9 \cdot 8 \equiv 0$, yet $8 \not\equiv 0$. OTOHand, even though $5 \cdot 24 \equiv 0$, this doesn't show that 5 is a \mathbb{Z}_{12} -ZD, since $24 \equiv 0$.]

An **integral domain** Γ is a **commutative** ring with no ZDs except for $\mathbf{0}_\Gamma$, the **trivial ZD**. If the characteristic of an **integral domain** is finite, then $\text{Char}(\Gamma)$ is a prime number. In particular, this holds if Γ is a field.

1: Fact. *If Γ is a field of finite order (finite cardinality) then $|\Gamma| = p^k$ for some prime p and posint k . Conversely, for each such prime p and $k \in \mathbb{Z}_+$, there exists a field of order p^k , and this field is unique up to field-isomorphism.* \diamond

Partial proof. For the prime $p := \text{Char}(\Gamma)$, there is a copy of \mathbb{Z}_p inside Γ , making Γ a \mathbb{Z}_p -vectorspace. Letting k be the dimension of this vectorspace, then, we obtain $|\Gamma| = p^k$.

The remaining FACTS take a fair amount of work to prove. \diamond

Totally-Ordered Sets. A **TOS** (Γ, \prec) has an antireflexive, transitive relation \prec so that for each $\alpha \neq \beta$ in Γ , either $\alpha \prec \beta$ or $\alpha \succ \beta$.

A subset $S \subset \Gamma$ is "**order-dense** in Γ " if:

For each pair $\alpha \prec \beta$ of elements in Γ , there exists $\tau \in S$ with $\alpha \prec \tau \prec \beta$.

If S is order-dense as a subset of *itself*, then say that " S is **order self-dense**". [E.g, TOS $(\mathbb{Q}, <)$ is order self-dense, but $(\mathbb{Z}, <)$ is not.]

Least upper-bound property [LUBP]. In TOS (Γ, \prec) , consider sets $A, B \subset \Gamma$ and a point $\gamma \in \Gamma$. Let

2.1:

$$A \preceq \gamma \quad \text{mean} \quad [\forall \alpha \in A, \text{ necessarily } \alpha \preceq \gamma];$$

$$A \preceq B \quad \text{mean} \quad [\forall \alpha \in A \text{ and } \forall \beta \in B: \alpha \preceq \beta].$$

An **upper-bound** for a set $A \subset \Gamma$ is an element $\gamma \in \Gamma$ such that $A \preceq \gamma$. Use $\text{UB}_\Gamma(A)$ for the *set* of upper-bnds, and $\text{LB}_\Gamma(A)$ for the lower-bnd-set. (Dispense with the subscript if clear from context.) Our (Γ, \prec) has the **LUBP** if:

Each non-void $A \subset \Gamma$ which is upper-bnded

2.2: *[i.e. $\text{UB}_\Gamma(A) \neq \emptyset$] has a least upper-bound. That is, $\text{UB}_\Gamma(A)$ has a minimum element.*

Reversing the inequalities yields the **greatest lower-bound property**, abbreviated **GLBP**.

The LUB of a set A (when it *has* a LUB!) is called the **supremum** of the set, and is written $\text{sup}(A)$ or $\text{sup}_\Gamma(A)$. Similarly, the **infimum** is the GLB, written $\text{inf}(A)$.

2.3: LUBP theorem. *TOS (Γ, \prec) has the LUBP IFF it has the GLBP.* \diamond

Proof of [LUBP \Rightarrow GLBP]. Fix a non-void lower-bnded subset $B \subset \Gamma$; so $A := \text{LB}_\Gamma(B)$ is non-empty. My goal is to produce a (hence the) greatest lower-bound for B , using that

$$\dagger: \quad A \stackrel{\text{def}}{=} \text{LB}_\Gamma(B), \quad \text{and}$$

$$\ddagger: \quad \text{UB}_\Gamma(A) \supset B.$$

Since $\text{UB}_\Gamma(A) \supset B \neq \emptyset$, and A is non-void, the LUBP applies, and tells us that $\lambda := \text{sup}_\Gamma(A)$ exists. In particular

$$\dagger': \quad \lambda \succcurlyeq A.$$

Since λ is the *least* upper-bnd, $\lambda \preccurlyeq \text{UB}_\Gamma(A) \supset B$ and so $\lambda \preccurlyeq B$. Restating, λ is a lower-bound of B . (Note: λ might or might not be in B .)

And, by (\dagger) and (\dagger') , this λ dominates each lower-bound of B . So λ is a *greatest* lower-bound of B . \blacklozenge

Making a Real Assumption. A TOS $(\Gamma, <)$ satisfying LUBP [equivalently, GLBP] is said to be **order-complete**. We take as an axiom [or derive via *Dedekind cuts* or *Cauchy sequences*] that

2.4: $(\mathbb{R}, <)$ is order-complete.

This means that the extended reals, $\overline{\mathbb{R}}$, satisfies a slightly stronger property: *Each*^{♥1} subset $A \subset \overline{\mathbb{R}}$ has a $\text{sup}(A)$ and an $\text{inf}(A)$ in $\overline{\mathbb{R}}$. In consequence, $\text{sup}()$ and $\text{inf}()$ are maps from the full $\mathcal{P}(\overline{\mathbb{R}}) \rightarrow \overline{\mathbb{R}}$.

Ordered-fields

An **ordered-field** $(\Gamma, +, \mathbf{0}, \cdot, \mathbf{1}; <)$ is a field which is an ordered set satisfying $\forall \alpha, \beta, \tau \in \Gamma$:

i: If $\alpha < \beta$ then $\alpha + \tau < \beta + \tau$. That is, relation “ $<$ ” is **translation invariant**.

ii: If $\alpha, \beta > \mathbf{0}$ then their product $\alpha \cdot \beta > \mathbf{0}$. I.e: “*Product is Positivity-Preserving*”.

3: Ordered-field lemma. In an ordered-field Γ :

a: If $\alpha \neq \mathbf{0}$, then $[\alpha > \mathbf{0}] \Leftrightarrow [-\alpha < \mathbf{0}]$.

b: Fix $\alpha > \beta$. If $\mu > \mathbf{0}$ then $\mu\alpha > \mu\beta$. If $\mu < \mathbf{0}$ then $\mu\alpha < \mu\beta$. Also, if $\mu \geq \mathbf{0}$ then $[\alpha \geq \beta] \Rightarrow [\mu\alpha \geq \mu\beta]$. If $\mu \leq \mathbf{0}$ then $[\alpha \geq \beta] \Rightarrow [\mu\alpha \leq \mu\beta]$.

Now suppose $\alpha_j > \beta_j > \mathbf{0}$, for $j \in \{1, 2\}$. Then $\alpha_1\alpha_2 > \beta_1\beta_2 > \mathbf{0}$.

c: For each $\alpha \neq \mathbf{0}$, necessarily $\alpha^2 > \mathbf{0}$. Hence $\mathbf{1} > \mathbf{0}$.

^{♥1}E.g, $\text{sup}(\emptyset) = -\infty$ and $\text{inf}(\emptyset) = +\infty$. Indeed, for $A \subset \overline{\mathbb{R}}$: $[A \neq \emptyset] \iff [\text{inf}(A) \preccurlyeq \text{sup}(A)]$.

d: $\text{Char}(\Gamma) = \infty$.

e: If $\mathbf{0} < \alpha < \beta$, then $\mathbf{0} < 1/\beta < 1/\alpha$. \blacklozenge

[**Note to self:** All but (e) holds for an *ordered integral-domain*. Adapt this material for that generalization, then do the *field-of-quotients* construction.]

Proof of (a). By translation invariance, if $\alpha > \mathbf{0}$ then $\alpha - \alpha > \mathbf{0} - \alpha$; etc.

Pf of (b). Saying $\alpha > \beta$ means $\alpha - \beta > \mathbf{0}$, hence (ii) implies $\mu[\alpha - \beta] > \mathbf{0}$, etc. Or if μ is negative, then $[-\mu][\alpha - \beta] > \mathbf{0}$; so now (a) and associativity of mult (and that $[-\mu] = \mu$) together imply that $\mu[\alpha - \beta] < \mathbf{0}$.

The two versions with *non-strict* inequality, “ \geq ”, follow from the strict inequalities.

Lastly, suppose $\alpha_j > \beta_j > \mathbf{0}$. Multiplying the 2nd by β_1 gives $\beta_1\alpha_2 > \beta_1\beta_2$. Multiplying the 1st by α_2 gives $\alpha_1\alpha_2 > \beta_1\alpha_2$. Transitivity yields $\alpha_1\alpha_2 > \beta_1\beta_2$. \blacklozenge

Pf of (c). If $\mathbf{1}$ negative, then (b) implies that $\mathbf{1} \cdot \mathbf{1}$ is positive; a contraction. And $\mathbf{1} \neq \mathbf{0}$, by the axioms for a field. Thus *trichotomy* forces that $\mathbf{1}$ is positive. \blacklozenge

Pf of (d). [I temporarily rename $\mathbf{0}$ to $\mathbf{0}_\Gamma$ and $\mathbf{1}$ to $\mathbf{1}_\Gamma$.] By (b), we now know that $\mathbf{0}_\Gamma < \mathbf{1}_\Gamma$. Since “ $<$ ” is translation-invariant, induction implies that for each posint n , the sum $\mathbf{1}_\Gamma + \mathbf{1}_\Gamma + \dots + \mathbf{1}_\Gamma$ is positive. \blacklozenge

Pf of (e). For $\gamma > \mathbf{0}$, its mult-inverse $1/\gamma$ is not $\mathbf{0}$ (since *it* has a mult-inv). Were $1/\gamma < \mathbf{0}$, then (b) implies $\gamma \cdot \frac{1}{\gamma} \stackrel{\text{note}}{=} \mathbf{1}$ is negative, contradicting (c). Hence

$$\gamma > \mathbf{0} \text{ implies } 1/\gamma > \mathbf{0}.$$

By (ii), product $\alpha\beta$ is positive, so $\frac{1}{\alpha\beta} > \mathbf{0}$. Multiplying the given $\mathbf{0} < \alpha < \beta$ by $\frac{1}{\alpha\beta}$ yields, courtesy (b), that $\mathbf{0} < 1/\beta < 1/\alpha$. \blacklozenge

4: Defn. For $n \in \mathbb{N}$ and $\beta \in \Gamma$, denote $\beta + \cdot^n \cdot + \beta$ by $n\beta$. For $n \in \mathbb{Z}_-$, use $n\beta$ for $[-\beta] + \cdot^n \cdot + [-\beta]$. Use \mathbb{Z}_Γ for $\{n \cdot \mathbf{1}_\Gamma \mid n \in \mathbb{Z}\}$, etc. \square

5: Exer. E1. Fix a field $(\Gamma, +, \mathbf{0}, \cdot, \mathbf{1})$. For $n \in \mathbb{Z}$ let \widehat{n} denote $n \cdot \mathbf{1}$ in Γ , as defined in (4).

i: For integers n_j and $d_j \in \mathbb{Z}$ with each $\widehat{d}_j \neq \mathbf{0}$, prove: If $\frac{n_1}{d_1} = \frac{n_2}{d_2}$ then $\widehat{n}_1/\widehat{d}_1 = \widehat{n}_2/\widehat{d}_2$.

ii: Suppose $p := \text{Char}(\Gamma)$ is finite, hence prime. Prove $n \mapsto \widehat{n}$ is a ring-homomorphism of \mathbb{Z} into Γ . Show that $n \mapsto \widehat{n}$ can be interpreted as an injective field-homomorphism of $\mathbb{Z}_p \hookrightarrow \Gamma$.

[For two fields \mathbf{F} and \mathbf{G} , is it possible to have a non-injective field-hom $\mathbf{F} \rightarrow \mathbf{G}$?]

iii: Now suppose $\text{Char}(\Gamma) = \infty$. Argue that for each $q \in \mathbb{Q}$, value $\widehat{q} \in \Gamma$ is well-defined by choosing integers $d \neq 0$ and n st. $q = \frac{n}{d}$, and then defining $\widehat{q} := \widehat{n}/\widehat{d}$.

iv: Suppose $\text{Char}(\Gamma) = \infty$, Prove that $q \xrightarrow{f} \widehat{q}$ is an injective field-homomorphism of $\mathbb{Q} \hookrightarrow \Gamma$. Further, if $(\Gamma, <)$ is an ordered field, then f is order-preserving as a map $(\mathbb{Q}, <) \hookrightarrow (\Gamma, <)$. \square

Archimedean fields

An ordered-field Γ is **Archimedean** if for each $\tau \in \Gamma$ there exists a natnum n with $n\mathbf{1}_\Gamma \geq \tau$. By setting $k := n+1$, we see this is equivalent to: $\exists k \in \mathbb{N}$ with $k\mathbf{1}_\Gamma > \tau$. Equivalently, $\text{UB}_\Gamma(\mathbb{Z}_\Gamma)$ is empty.

6.1: Archy lemma. If ordered-field $(\Gamma, +, \mathbf{0}_\Gamma, \cdot, \mathbf{1}_\Gamma; <)$ is Archimedean, then for each $\beta \succ \mathbf{0}_\Gamma$: The upper-bound set of $M_\beta := \{n\beta \mid n \in \mathbb{N}\}$ is empty.

Moreover, if there exists a $\beta \in \Gamma$ with $\text{UB}_\Gamma(M_\beta)$ empty, then Γ is Archimedean. \diamond

Pf. Fix a posint K with $K\mathbf{1}_\Gamma \geq \mathbf{1}_\Gamma/\beta \stackrel{\text{note}}{>} \mathbf{0}$. Multiply by β to conclude that $K\beta \geq \mathbf{1}_\Gamma$, by (3). So for each natnum n , element $[nK] \cdot \beta$ dominates $n\mathbf{1}_\Gamma$. Thus $\text{UB}_\Gamma(M_\beta) \subset \text{UB}_\Gamma(\mathbb{Z}_\Gamma)$. And $\text{UB}_\Gamma(\mathbb{Z}_\Gamma) = \emptyset$.

Exercise E2: The converse is left the Reader. \diamond

6.2: Corollary. Suppose ordered-field $(\Gamma, +, \mathbf{0}_\Gamma, \cdot, \mathbf{1}_\Gamma; <)$ is Archimedean. Then $\text{UB}_\Gamma(\mathbb{Z}_\Gamma) = \emptyset = \text{LB}_\Gamma(\mathbb{Z}_\Gamma)$. Moreover, for each $\alpha \in \Gamma$, there exists a unique integer K with $[K-1]\mathbf{1}_\Gamma \leq \alpha < K\mathbf{1}_\Gamma$. \diamond

Proof. The order-reversing map $x \mapsto -x$ on Γ sends lower-bnd-sets to upper-bnd-sets, etc., hence $\text{LB}(\mathbb{Z}_\Gamma)$ is empty.

Setting $U_k := \{x \in \Gamma \mid x \geq k\mathbf{1}_\Gamma\}$, the foregoing tell us that $\bigcup_{k \in \mathbb{Z}} U_k$ is Γ , and $\bigcap_{k \in \mathbb{Z}} U_k$ is empty. These sets are nested, $\dots, U_{-1} \supset U_0 \supset U_1 \supset U_2 \supset \dots$ so there is a unique integer $K \in \mathbb{Z}$ with the given α in the difference-set $U_{K-1} \setminus U_K$. \blacklozenge

7: OC \Rightarrow Archimedean theorem. Suppose ordered-field $(\Gamma, +, \mathbf{0}_\Gamma, \cdot, \mathbf{1}_\Gamma; <)$ is order-complete. Then Γ is Archimedean. \diamond

Proof. FTSOContradiction, suppose $\text{UB}(\mathbb{Z}_\Gamma)$ is not empty; so \mathbb{Z}_Γ has a Γ -LUB; call it τ . Now $\tau - \mathbf{1}_\Gamma$ is less than τ , hence cannot upper-bnd \mathbb{Z}_Γ . Consequently, $\exists n \in \mathbb{Z}_+$ with $[n-1]\mathbf{1}_\Gamma > \tau - \mathbf{1}_\Gamma$. Thus $n\mathbf{1}_\Gamma > \tau$, which is a blatant contradiction. \blacklozenge

8.0: Order-dense lemma. Fix $(\Gamma, +, \widehat{0}, \cdot, \widehat{1}; <)$, an Archimedean field. For $q \in \mathbb{Q}$, define $\widehat{q} \in \Gamma$ as in (5), and let $\widehat{\mathbb{Q}}$ denote the copy of \mathbb{Q} inside Γ . Then $\widehat{\mathbb{Q}}$ is order-dense in Γ . \diamond

Proof. Fix $\alpha < \beta$ in Γ . We will produce integers $N \in \mathbb{Z}_+$ and K such that

$$*: \quad \alpha < \widehat{K/N} < \beta.$$

By (6.1) there is posint N with $N[\beta - \alpha] > \widehat{2}$. Dropping the “ $\widehat{}$ ” symbol for the rest of the proof, we have $\frac{\beta - \alpha}{2} > \frac{1}{N}$. Hence $\alpha + \frac{1}{N} < \alpha + \frac{\beta - \alpha}{2}$, so

$$**: \quad \alpha + \frac{1}{N} < \beta.$$

Courtesy (6.2), $\exists K \in \mathbb{Z}$ with $[K-1] \leq N\alpha < K$, i.e. $N\alpha < K \leq [N\alpha] + 1$. Hence $\alpha < \frac{K}{N} \leq \alpha + \frac{1}{N}$, since N is positive. This and (**), yield (*). \blacklozenge

Complete ordered-field(s)

We now come to the main result.

9: Theorem. Suppose $(\Gamma, +, \widehat{0}, \cdot, \widehat{1}; <)$ and $(\mathbf{F}, +, \overline{0}, \cdot, \overline{1}; <)$ are ordered-fields. Then they are ordered-field-isomorphic. Moreover, there is a unique OF-isomorphism between them. \diamond

Proof (sketch). For $S \subset \mathbb{Q}$, let $\widehat{S} := \{\widehat{q} \mid q \in S\} \stackrel{\text{note}}{\subset} \Gamma$. Define similarly $\overline{S} \subset \mathbf{F}$. For $\alpha \in \Gamma$ and $x \in \mathbf{F}$, define

$$U_\alpha := \{q \in \mathbb{Q} \mid \widehat{q} \leq \alpha\} \quad \text{and} \quad V_x := \{q \in \mathbb{Q} \mid \overline{q} \leq x\}.$$

There exist $q, r \in \mathbb{Q}$ with $\alpha - \widehat{1} < q < \alpha < r < \alpha + \widehat{1}$; this, by (8.0), density. Hence U_α is non-void and upper-bnded in \mathbb{Q} , so $\overline{U_\alpha}$ is non-void and upper-bnded in \mathbf{F} . I.e, $\sup_{\mathbf{F}}(\overline{U_\alpha})$ exists in \mathbf{F} . Consequently, we have a well-defined map $\Phi: \Gamma \rightarrow \mathbf{F}$, by

$$\Phi(\alpha) := \sup_{\mathbf{F}}(\overline{U_\alpha}).$$

Evidently Φ is *weakly* order-preserving in that for all $\alpha, \beta \in \Gamma$: $[\alpha \leq \beta] \Rightarrow [\Phi(\alpha) \leq \Phi(\beta)]$.

Similarly, $\mathcal{G}(x) := \sup_{\Gamma}(\widehat{V_x})$ is a weakly-OP map $\mathcal{G}: \mathbf{F} \rightarrow \Gamma$. \diamond

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