There is one order-complete ordered-field

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Rings. In a ring \((\Gamma,+,0_{\Gamma},\ast,1_{\Gamma})\), if there is a posint \(n\) so that \(1_{\Gamma} + 1_{\Gamma} + \ldots + 1_{\Gamma} = 0_{\Gamma}\), then the smallest \(n\) is called the characteristic of the ring, and I write \(\text{Char}(\Gamma) = n\). If no such posint exists, then I will write \(\text{Char}(\Gamma) = \infty\); however, the standard term is \(\text{Char}(\Gamma) = 0\), and you will see this in algebra texts and in some of my notes.

A ring is commutative (abbrev., comm-ring) if its multiplication is commutative. In a comm-ring \(\Gamma\), a zero-divisor \(\alpha \in \Gamma\) admits a non-zero elt \(\beta \in \Gamma\) (this \(\beta\) need not be unique) so that \(\alpha \beta = 0_{\Gamma}\). Use \(\mathbb{Z}\D\) for zero-divisor. [Letting \(\equiv\) denote \(\equiv_{12}\), in the \(\mathbb{Z}_{12}\) ring, \(9\) is a \(\mathbb{Z}\D\), since \(9 \cdot 8 \equiv 0\), yet \(8 \not\equiv 0\). OTOH, even though \(5 \cdot 24 \equiv 0\), this doesn’t show that \(5\) is a \(\mathbb{Z}_{12}\D\), since \(24 \equiv 0\).]

An integral domain \(\Gamma\) is a commutative ring with no \(\mathbb{Z}\Ds\) except for \(0_{\Gamma}\), the trivial \(\mathbb{Z}\D\). If the characteristic of an integral domain is finite, then \(\text{Char}(\Gamma)\) is a prime number. In particular, this holds if \(\Gamma\) is a field.

1: Fact. If \(\Gamma\) is a field of finite order (finite cardinality) then \(|\Gamma| = p^{k}\) for some prime \(p\) and posint \(k\). Conversely, for each such prime \(p\) and \(k \in \mathbb{Z}_{+}\), there exists a field of order \(p^{k}\), and this field is unique up to field-isomorphism.

Partial proof. For the prime \(p := \text{Char}(\Gamma)\), there is a copy of \(\mathbb{Z}_{p}\) inside \(\Gamma\), making \(\Gamma\) a \(\mathbb{Z}_{p}\)-vectorspace. Letting \(k\) be the dimension of this vectorspace, then, we obtain \(|\Gamma| = p^{k}|\).

The remaining Facts take a fair amount of work to prove.

Totally-Ordered Sets. A TOS \((\Gamma,\prec)\) has an antireflexive, transitive relation \(\prec\) so that for each \(\alpha \neq \beta\) in \(\Gamma\), either \(\alpha \prec \beta\) or \(\alpha \succ \beta\).

A subset \(S \subset \Gamma\) is “order-dense in \(\Gamma\)” if:

- For each pair \(\alpha \prec \beta\) of elements in \(\Gamma\), there exists \(\tau \in S\) with \(\alpha \prec \tau \prec \beta\).

If \(S\) is order-dense as a subset of itself, then say that “\(S\) is order self-dense,” [E.g., TOS \((\mathbb{Q},\prec)\) is order self-dense, but \((\mathbb{Z},\prec)\) is not.]

Least upper-bound property [LUBP]. In TOS \((\Gamma,\prec)\), consider sets \(A,B \subset \Gamma\) and a point \(\gamma \in \Gamma\). Let

\[\begin{align*}
A \preceq \gamma & \quad \text{mean \quad } \{\forall \alpha \in A, \text{ necessarily } \alpha \preceq \gamma\}; \\
A \preceq B & \quad \text{mean \quad } \{\forall \alpha \in A \text{ and } \forall \beta \in B: \alpha \preceq \beta\}.
\end{align*}\]

An upper-bound for a set \(A \subset \Gamma\) is an element \(\gamma \in \Gamma\) such that \(A \preceq \gamma\). Use \(\text{UB}_{\Gamma}(A)\) for the set of upper-bounds, and \(\text{LB}_{\Gamma}(A)\) for the lower-bnd–set. (Dispense with the subscript if clear from context.) Our \((\Gamma,\prec)\) has the LUBP if:

- Each non-void \(A \subset \Gamma\) which is upper-bnded \(\Rightarrow \) has a least upper-bound. That is, \(\text{UB}_{\Gamma}(A)\) has a minimum element.

Reversing the inequalities yields the greatest lower-bound property, abbreviated GLBP.

The LUB of a set \(A\) (when it has a LUB!) is called the supremum of the set, and is written \(\text{sup}(A)\) or \(\text{sup}_{\Gamma}(A)\). Similarly, the infimum is the GLB, written \(\text{inf}(A)\).

2.3: LUBP theorem. TOS \((\Gamma,\prec)\) has the LUBP \IFF it has the GLBP.

Proof of [LUBP ⇒ GLBP]. Fix a non-void lower-bnded subset \(B \subset \Gamma\); so \(A := \text{LB}_{\Gamma}(B)\) is non-empty. My goal is to produce a (hence the) greatest lower-bound for \(B\), using that

\[\begin{align*}
\dagger: \quad A & \overset{\text{def}}{=} \text{LB}_{\Gamma}(B), \\
\ddagger: \quad \text{UB}_{\Gamma}(A) & \supset B.
\end{align*}\]
Since $\cup B_\Gamma(A) \supset B \neq \emptyset$, and $A$ is non-void, the LUBP applies, and tells us that $\lambda := \sup_\Gamma(A)$ exists. In particular

\[ \lambda \succ A. \]

Since $\lambda$ is the least upper-bnd, $\lambda \leq \cup B_\Gamma(A) \supset B$ and so $\lambda \leq B$. Restating, $\lambda$ is a lower-bound of $B$. (Note: $\lambda$ might or might not be in $B$.)

And, by (†) and (†′), this $\lambda$ dominates each lower-bound of $B$. So $\lambda$ is a greatest lower-bound of $B$. ♦

**Making a Real Assumption.** A TOS $(\Gamma, <)$ satisfying LUBP [equivalently, GLBP] is said to be order-complete. We take as an axiom [or derive via Dedekind cuts or Cauchy sequences] that

2.4: $(\mathbb{R}, <)$ is order-complete.

This means that the extended reals, $\mathbb{R}$, satisfies a slightly stronger property: Each \footnote{E.g., $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = +\infty$. Indeed, for $A \subset \mathbb{R}$: $[A \neq \emptyset] \iff [\inf(A) \leq \sup(A)]$.} subset $A \subset \mathbb{R}$ has a $\sup(A)$ and an $\inf(A)$ in $\mathbb{R}$. In consequence, $\sup()$ and $\inf()$ are maps from the full $\mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$.

**Ordered-fields**

An ordered-field $(\Gamma, +, 0, \cdot, 1; <)$ is a field which is an ordered set satisfying $\forall \alpha, \beta, \tau \in \Gamma$:

i: If $\alpha < \beta$ then $\alpha + \tau < \beta + \tau$. That is, relation “$<$” is translation invariant.

ii: If $\alpha, \beta > 0$ then their product $\alpha \beta > 0$. I.e.: “Product is Positivity-Preserving”.

3: Ordered-field lemma. In an ordered-field $\Gamma$:

a: If $\alpha \neq 0$, then $[\alpha > 0] \Leftrightarrow [-\alpha < 0]$.

b: Fix $\alpha \succ \beta$. If $\mu > 0$ then $\mu \alpha > \mu \beta$. If $\mu < 0$ then $\mu \alpha < \mu \beta$. Also, if $\mu \geq 0$ then $[\alpha \geq \beta] \Rightarrow [\mu \alpha \geq \mu \beta]$.

If $\mu \leq 0$ then $[\alpha \geq \beta] \Rightarrow [\mu \alpha \leq \mu \beta]$.

Now suppose $\alpha_j > \beta_j > 0$, for $j \in \{1, 2\}$. Then $\alpha_1 \alpha_2 > \beta_1 \beta_2 > 0$.

c: For each $\alpha \neq 0$, necessarily $\alpha^2 > 0$. Hence $1 > 0$.

d: $\text{Char}(\Gamma) = \infty$.

e: If $0 < \alpha < \beta$, then $0 < 1/\beta < 1/\alpha$. ♦

[ Note to self: All but (e) holds for an ordered integral-domain. Adapt this material for that generalization, then do the field-of-quotiens construction. ]

**Proof of (a).** By translation invariance, if $\alpha > 0$ then $\alpha - \alpha > 0 - \alpha$; etc.

**Pf of (b).** Saying $\alpha \succ \beta$ means $\alpha - \beta > 0$, hence (ii) implies $\mu[\alpha - \beta] > 0$, etc. Or if $\mu$ is negative, then $[-\mu][\alpha - \beta] > 0$; so now (a) and associativity of mult (and that $[-\mu] = \mu$) together imply that $\mu[\alpha - \beta] < 0$.

The two versions with non-strict inequality, “$\geq$”, follow from the strict inequalities.

Lastly, suppose $\alpha_j > \beta_j > 0$. Multiplying the $2^\text{nd}$ by $\beta_1$ gives $\beta_1 \alpha_2 > \beta_1 \beta_2$. Multiplying the $1^\text{st}$ by $\alpha_2$ gives $\alpha_1 \alpha_2 > \beta_1 \beta_2$. Transitivity yields $\alpha_1 \alpha_2 > \beta_1 \beta_2$. ♦

**Pf of (c).** If 1 negative, then (b) implies that $1 \cdot 1$ is positive; a contraction. And $1 \neq 0$, by the axioms for a field. Thus trichotomy forces that 1 is positive. ♦

**Pf of (d).** [I temporarily rename 0 to $0_\Gamma$ and 1 to $1_\Gamma$.] By (b), we now know that $0_\Gamma < 1_\Gamma$. Since “$<$” is translation-invariant, induction implies that for each point $n$, the sum $1_\Gamma + 1_\Gamma + \cdots + 1_\Gamma$ is positive. ♦

**Pf of (e).** For $\gamma > 0$, its mult-inverse $1/\gamma$ is not 0 (since it has a mult-inv). Were $1/\gamma < 0$, then (b) implies $\gamma \cdot 1/\gamma \not= 1$ is negative, contradicting (c). Hence $\gamma > 0$ implies $1/\gamma > 0$.

By (ii), product $\alpha \beta$ is positive, so $1/\alpha \beta > 0$. Multiplying the given $0 < \alpha < \beta$ by $1/\alpha \beta$ yields, courtesy (b), that $0 < 1/\beta < 1/\alpha$. ♦
4: **Defn.** For $n \in \mathbb{N}$ and $\beta \in \Gamma$, denote $\beta + \frac{n}{\beta} + \beta$. For $n \in \mathbb{Z}$, use $n\beta$ for $[-\beta] + \tau^0 + [-\beta]$. Use $\mathbb{Z}_\Gamma$ for $\{n1_\Gamma \mid n \in \mathbb{Z}\}$, etc. \hfill \square

5: **Exer. E1.** Fix a field $(\Gamma, +, 0, \cdot, 1)$. For $n \in \mathbb{Z}$ let $\hat{n}$ denote $n1_\Gamma$ in $\Gamma$, as defined in (4).

\[ i: \text{For integers } n_j \text{ and } d_j \in \mathbb{Z} \text{ with each } \hat{d}_j \neq 0, \text{ prove: If } \frac{n_1}{d_1} = \frac{n_2}{d_2} \text{ then } \hat{n}_1/\hat{d}_1 = \hat{n}_2/\hat{d}_2. \]

\[ ii: \text{Suppose } p := \text{Char}(\Gamma) \text{ is finite, hence prime. Prove } n \mapsto \hat{n} \text{ is a ring-homomorphism of } \mathbb{Z} \text{ into } \Gamma. \text{ Show that } n \mapsto \hat{n} \text{ can be interpreted as an injective field-homomorphism of } \mathbb{Z}_p \rightarrow \Gamma. \]

\[ \text{[For two fields } F \text{ and } G, \text{ is it possible to have a non-injective field-homomorphism } F \rightarrow G?] \]

\[ iii: \text{Now suppose } \text{Char}(\Gamma) = \infty. \text{ Argue that for each } q \in \mathbb{Q}, \text{ value } \hat{q} \in \Gamma \text{ is well-defined by choosing integers } d \neq 0 \text{ and } n \text{ st. } q = \frac{n}{d}, \text{ and then defining } \hat{q} := \frac{\hat{n}}{\hat{d}}. \]

\[ iv: \text{Suppose } \text{Char}(\Gamma) = \infty. \text{ Prove that } q \mapsto \hat{q} \text{ is an injective field-homomorphism of } \mathbb{Q} \rightarrow \Gamma. \text{ Further, if } (\Gamma, <) \text{ is an ordered field, then } f \text{ is order-preserving as a map } (\mathbb{Q}, <) \rightarrow (\Gamma, <). \] \hfill \square

#### Archimedean fields

An ordered-field $\Gamma$ is **Archimedean** if for each $\tau \in \Gamma$ there exists a natnum $n$ with $n1_\Gamma \geq \tau$. By setting $k := n+1$, we see this is equivalent to: $\exists k \in \mathbb{N}$ with $k1_\Gamma \geq \tau$. Equivalently, $\text{UB}_F(\mathbb{Z}_\Gamma)$ is empty.

6.1: **Archy lemma.** If ordered-field $(\Gamma, +, 0_\Gamma, \cdot, 1_\Gamma; <)$ is Archimedean, then for each $\beta > 0_\Gamma$: The upper-bound set of $M_\beta := \{n\beta \mid n \in \mathbb{N}\}$ is empty. Moreover, if there exists a $\beta \in \Gamma$ with $\text{UB}_F(M_\beta)$ empty, then $\Gamma$ is Archimedean. \hfill \diamond

**Pf.** Fix a posint $K$ with $K1_\Gamma \geq 1_\Gamma/\beta$ note. Multiply by $\beta$ to conclude that $K\beta \geq 1_\Gamma$, by (3). So for each natnum $n$, element $[nK] \cdot \beta$ dominates $n1_\Gamma$. Thus $\text{UB}_F(M_\beta) \subset \text{UB}_F(\mathbb{Z}_\Gamma)$. And $\text{UB}_F(\mathbb{Z}_\Gamma) = \emptyset$. \hfill \diamond

Exercise E2: The converse is left the Reader. \hfill \diamond

6.2: **Corollary.** Suppose ordered-field $(\Gamma, +, 0_\Gamma, \cdot, 1_\Gamma; <)$ is Archimedean. Then $\text{UB}_F(\mathbb{Z}_\Gamma) = \emptyset = \text{LB}_F(\mathbb{Z}_\Gamma)$. Moreover, for each $\alpha \in \Gamma$, there exists a unique integer $K$ with $[K-1]1_\Gamma \leq \alpha < K1_\Gamma$. \hfill \diamond

**Proof.** The order-reversing map $x \mapsto -x$ on $\Gamma$ sends lower-bnd-sets to upper-bnd-sets, etc., hence $\text{LB}(\mathbb{Z}_\Gamma)$ is empty.

Setting $U_k := \{x \in \Gamma \mid x \geq k1_\Gamma\}$, the foregoing tell us that $\bigcup_{k \in \mathbb{Z}} U_k$ is $\Gamma$, and $\bigcap_{k \in \mathbb{Z}} U_k$ is empty. These sets are nested, $\ldots, U_1 \supset U_0 \supset U_1 \supset U_2 \supset \ldots$ so there is a unique integer $K \in \mathbb{Z}$ with the given $\alpha$ in the difference-set $U_{K-1} \smallsetminus U_K$. \hfill \diamond

7: **OC⇒Archimedean theorem.** Suppose ordered-field $(\Gamma, +, 0_\Gamma, \cdot, 1_\Gamma; <)$ is order-complete. Then $\Gamma$ is Archimedean. \hfill \diamond

**Proof.** By contradiction, suppose $\text{UB}(\mathbb{Z}_\Gamma)$ is not empty; so $\mathbb{Z}_\Gamma$ has a $\Gamma$–LUB; call it $\tau$. Now $\tau - 1_\Gamma$ is less than $\tau$, hence cannot upper-bnd $\mathbb{Z}_\Gamma$. Consequently, $\exists n \in \mathbb{Z}_+$ with $[n-1]1_\Gamma > \tau - 1_\Gamma$. Thus $n1_\Gamma > \tau$, which is a blatant contradiction. \hfill \diamond

8.0: **Order-dense lemma.** Fix $(\Gamma, +, 0_\Gamma, \cdot, 1_\Gamma; <)$, an Archimedean field. For $q \in \mathbb{Q}$, define $\hat{q} \in \Gamma$ as in (5), and let $\hat{\mathbb{Q}}$ denote the copy of $\mathbb{Q}$ inside $\Gamma$. Then $\hat{\mathbb{Q}}$ is order-dense in $\Gamma$. \hfill \diamond

**Proof.** Fix $\alpha < \beta$ in $\Gamma$. We will produce integers $N \in \mathbb{Z}_+$ and $K$ such that $\ast: \alpha < \frac{K}{N} < \beta.$

By (6.1) there is posint $N$ with $N[\beta - \alpha] > \frac{1}{\beta}$. Dropping the “$^\ast$” symbol for the rest of the proof, we have $\frac{\beta - \alpha}{2} > \frac{1}{N}$. Hence $\alpha + \frac{1}{N} < \alpha + \frac{\beta - \alpha}{2}$, so $\ast\ast: \alpha + \frac{1}{N} < \beta.$

Courtesy (6.2), $\exists K \in \mathbb{Z}$ with $[K-1] \leq N\alpha < K$, i.e., $N\alpha < K \leq [N\alpha] + 1$. Hence $\alpha < \frac{K}{N} \leq \alpha + \frac{1}{N}$, since $N$ is positive. This and (\ast\ast), yield (\ast). \hfill \diamond
Complete ordered-field(s)

We now come to the main result.

**9: Theorem.** Suppose \((\Gamma, +, \hat{0}, \cdot, \hat{1}; <)\) and \((F, +, \hat{0}, \cdot, \hat{1}; <)\) are ordered-fields. Then they are ordered-field-isomorphic. Moreover, there is a unique OF-isomorphism between them. 

\(\diamondsuit\)

**Proof (sketch).** For \(S \subset \mathbb{Q}\), let \(\hat{S} := \{\hat{q} \mid q \in S\} \subset \Gamma\). Define similarly \(\hat{S} \subset F\). For \(\alpha \in \Gamma\) and \(x \in F\), define

\[ U_\alpha := \{q \in \mathbb{Q} \mid \hat{q} \leq \alpha\} \quad \text{and} \quad V_x := \{q \in \mathbb{Q} \mid \hat{q} \leq x\}. \]

There exist \(q, r \in \mathbb{Q}\) with \(\alpha - \hat{1} < q < \alpha < r < \alpha + \hat{1}\); this, by (8.0), density. Hence \(U_\alpha\) is non-void and upper-bnded in \(\mathbb{Q}\), so \(\sup F(U_\alpha)\) exists in \(F\). Consequently, we have a well-defined map \(\Phi: \Gamma \to F\), by

\[ \Phi(\alpha) := \sup F(U_\alpha). \]

Evidently \(\Phi\) is weakly order-preserving in that for all \(\alpha, \beta \in \Gamma\): \([\alpha \leq \beta] \Rightarrow [\Phi(\alpha) \leq \Phi(\beta)]\).

Similarly, \(G(x) := \sup F(V_x)\) is a weakly-OP map \(G: F \to \Gamma\).  

\(\diamondsuit\)